Revisiting Isbell's work on the Zassenhaus theorem

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The Refinement Theorem

Theorem (Zassenhaus-Schreier refinement)

Consider two normal series

$$G = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n = 0$$

and

$$G = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_m = 0.$$

There are refinements of the two series which are isomorphic (there is a bijection $\phi\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots m\}$ *such that* $X_{i-1}/X_i \approx Y_{\phi(i)-1}/Y_{\phi(i)}$*).*

- The isomorphism is canonical.
- There is at most one canonical isomorphism between two normal series.
- The constructed refinements (in the proof) are the coarsest refinements that are canonically isomorphic.

Subfactors and Projections

- "Zassenhaus Theorem Supercedes the Jordan-Hölder Theorem"- J. Isbell
- A *subfactor* of a group G is a pair of two subgroups (X, X^+) of G such that $X \lhd X^+$.
- 1. $X \subseteq X^+$,
  2. X^+ is a conormal subgroup of G, and,
  3. $\iota_{X^+}^{-1} X$ is normal in $X^+/1$.
- The *projection* of a subgroup Y in the subfactor (X, X^+) is defined as $(Y \wedge X^+) \vee X$.

Projection

A diagram

$$X_0 \overset{f_1}{\text{------}} X_1 \overset{f_2}{\text{------}} X_2 \overset{f_3}{\text{------}} X_3 \overset{f_4}{\text{------}} X_4 \cdots\cdots\cdots X_{n-1} \overset{f_n}{\text{------}} X_n$$

of groups and homomorphisms is called a *zigzag*. Note that the arrowheads have been deliberately left out as they are allowed to appear either on the left or on the right.

Notice that the projection of a subgroup Y in a subfactor (X, X^+) may be equivalently defined by chasing the subgroup Y from the group G to the other end and then back to G in the following zigzag:

$$G \xleftarrow{\iota_{X^+}} X^+ \xrightarrow{\pi_{\iota_{X^+}^{-1}X}} X^+/X.$$

Isbell's Projection Lemma

Lemma

1. *The projection of a subfactor* (Y, Y^+) *in a subfactor* (X, X^+) *is a subfactor* (M, M^+);
2. (M, M^+) *and* (X, X^+) *have the same projection in* (Y, Y^+).

Proof of Projection Lemma

$$
\begin{aligned}
((Y \wedge X^+) \vee X) \wedge Y^+ &= ((Y \wedge X^+) \vee X) \wedge Y^+ \wedge X^+ \\
&= \iota_{X^+}\iota_{X^+}^{-1}(((Y \wedge X^+) \vee X) \wedge Y^+) \\
&= \iota_{X^+}(\iota_{X^+}^{-1}((Y \wedge X^+) \vee X) \wedge \iota_{X^+}^{-1}(Y^+)) \\
&= \iota_{X^+}((\iota_{X^+}^{-1}(Y \wedge X^+) \vee \iota_{X^+}^{-1}(X)) \wedge \iota_{X^+}^{-1}(Y^+)) \\
&= \iota_{X^+}((\iota_{X^+}^{-1}(Y) \vee \iota_{X^+}^{-1}(X)) \wedge \iota_{X^+}^{-1}(Y^+)) \\
&= \iota_{X^+}((\iota_{X^+}^{-1}(Y)) \vee (\iota_{X^+}^{-1}(X) \wedge \iota_{X^+}^{-1}(Y^+))) \\
&= \iota_{X^+}((\iota_{X^+}^{-1}(Y)) \vee (\iota_{X^+}^{-1}(X \wedge Y^+))) \\
&= \iota_{X^+}((\iota_{X^+}^{-1}(Y))) \vee \iota_{X^+}(\iota_{X^+}^{-1}(X \wedge Y^+)) \\
&= (Y \wedge X^+) \vee ((X \wedge Y^+) \wedge X^+) \\
&= (Y \wedge X^+) \vee (X \wedge Y^+).
\end{aligned}
$$

Restricted Modular Law

Lemma (Restricted Modular Law [2])

For any three subgroups, X, Y, and Z of a group G, if either Y is normal and Z is conormal, or Y is conormal and X is normal, then we have:

$$X \subseteq Z \Rightarrow X \vee (Y \wedge Z) = (X \vee Y) \wedge Z.$$

Butterfly lemma

Lemma

(butterfly lemma) Given two subfactors (X, X^+) and (Y, Y^+) in a group G then:

1. *the projections (M, M^+) of (X, X^+) in (Y, Y^+) and (N, N^+) of (Y, Y^+) in (X, X^+) project onto each other;*
2. *N^+/N is isomorphic to M^+/M.*

$$Y^+/Y \xleftarrow{\pi_{\iota_{Y^+}^{-1} Y}} Y^+ \xrightarrow{\iota_{Y^+}} G \xleftarrow{\iota_{X^+}} X^+ \xrightarrow{\pi_{\iota_{X^+}^{-1} X}} X^+/X.$$

In the zigzag above, it follows that (Y, Y^+) projects onto (X, X^+) if and only if

1. chasing the smallest subgroup forward along the zigzag yields the smallest subgroup, and,
2. chasing the largest subgroup forward along the zigzag yields the largest subgroup.

Projection onto in terms of chasing

$$Y^+/Y \xleftarrow{\pi_{\iota_{Y^+}^{-1}Y}} Y^+ \xrightarrow{\iota_{Y^+}} G \xleftarrow{\iota_{X^+}} X^+ \xrightarrow{\pi_{\iota_{X^+}^{-1}X}} X^+/X.$$

- $\iota_{Y^+}\pi^{-1}_{\iota_{Y^+}^{-1}Y}0 = \iota_{Y^+}\iota_{Y^+}^{-1}Y = Y$
- $\iota_{X^+}\pi^{-1}_{\iota_{X^+}^{-1}X}\pi_{\iota_{X^+}^{-1}X}\iota_{X^+}^{-1}\iota_{Y^+}\pi^{-1}_{\iota_{Y^+}^{-1}Y}0 = X$
- $\iota_{X^+}^{-1}\iota_{X^+}\pi^{-1}_{\iota_{X^+}^{-1}X}\pi_{\iota_{X^+}^{-1}X}\iota_{X^+}^{-1}\iota_{Y^+}\pi^{-1}_{\iota_{Y^+}^{-1}Y}0 = \iota_{X^+}^{-1}X$
- $\pi_{\iota_{X^+}^{-1}X}\pi^{-1}_{\iota_{X^+}^{-1}X}\pi_{\iota_{X^+}^{-1}X}\iota_{X^+}^{-1}\iota_{Y^+}\pi^{-1}_{\iota_{Y^+}^{-1}Y}0 = \pi_{\iota_{X^+}^{-1}X}\iota_{X^+}^{-1}X$
- $\pi_{\iota_{X^+}^{-1}X}\iota_{X^+}^{-1}\iota_{Y^+}\pi^{-1}_{\iota_{Y^+}^{-1}Y}0 = 0$

Universal Isomorphism Theorem

Theorem (Universal Isomorphism Theorem [2])

If the largest and smallest subgroups are preserved by chasing them from each end node of the zigzag to the opposite end node, then it induces a homomorphism and the induced homomorphism is an isomorphism.

Projectively Isomorphic

Definition

Two normal series $G = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n = 0$ and $G = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m = 0$ in a group G are said to be *projectively isomorphic* if each subfactor (X_{i+1}, X_i) projects onto one of the subfactors (Y_{j+1}, Y_j) such that (Y_{j+1}, Y_j) projects onto (X_{i+1}, X_i).

Isbell's refinement theorem

Theorem (Isbell-Zassenhaus-Schreier refinement)

Consider two normal series

$$G = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n = 0$$

and

$$G = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_m = 0.$$

Then the projections of the Y_j *in all* (X_{i+1}, X_i) *and the projections of the* X_i *in all* (Y_{j+1}, Y_j) *form projectively isomorphic series refining the given series. Moreover, every subfactor* (X, X^+) *of* (X_{i+1}, X_i) *onto which some subfactor of* (Y_{j+1}, Y_j) *projects is smaller than the projection* (M, M^+) *of* (Y_{j+1}, Y_j) *in* (X_{i+1}, X_i)*. Therefore the refinements are the coarsest projectively isomorphic refinements.*

Jordan-Hölder theorem

Corollary (Jordan-Hölder theorem)

Consider two composition series

$$G = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n = 0$$

and

$$G = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_m = 0$$

of the same group G. Then the two composition series are projectively isomorphic.

Noetherian Form

A. Goswami and Z. Janelidze, *Duality in non-abelian algebra IV. Duality for groups and a universal isohomomorphism theorem*

- Abelian Categories: category of modules over a ring
- (G. Janelidze, L. Márki, W. Tholen) Semi-abelian categories: the category of groups, the category of graded abelian groups, the category of Lie algebras, the category of cocommutative Hopf algebras over an algebraically closed field of characteristic zero, the category of Heyting semilattices and the dual of the category of pointed sets.
- Grandis exact categories: the category of modular lattices and modular connections and the category of sets and partial bijections.

Duality

Statement	Dual Statement
G is a group	G is a group
S is a subgroup of G	S is a subgroup of G
$f\colon X \to Y$ is a homomorphism from the group X to the group Y	$f\colon Y \to X$ is a homomorphism from the group Y to the group X
for two subgroups S and T of G $S \subseteq T$ is an inclusion	for two subgroups S and T of G $T \subseteq S$ is an inclusion
for two subgroups A of X and B of Y under the homomorphism $f\colon X \to Y$ B is the direct image of A (i.e. $B = fA$)	for two subgroups A of X and B of Y under the homomorphism $f\colon Y \to X$ B is the inverse image of A (i.e. $B = f^{-1}A$)
the composite $h = gf$ of homomorphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is a homomorphism $h\colon X \to Z$	the composite $h = fg$ of homomorphisms $f\colon Y \to X$ and $g\colon Z \to Y$ is a homomorphism $h\colon Z \to X$

Dual Notions

Given subgroups $B \subseteq A$ of a group G. Then B is said to be conormal to A if B is normal and $\pi_B A$ is conormal. If B is conormal in A then (B, A) is called a *subcofactor* of G.

Projection in a subcofactor

The projection of a subgroup X in a subcofactor (B, A) is then defined as $(X \vee B) \wedge A$. This may also be seen as chasing the subgroup X forwards and then backwards along the following zigzag:

$$G \xrightarrow{\pi_B} G/B \xleftarrow{\iota_{\pi_B A}} \pi_B A.$$

Given a subcofactor (B, A), we may define $B \backslash A = \pi_B A$.

Conormal Series

Definition

A conormal series in a group G is a chain of subgroups

$$G = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m = 0$$

such that each subgroup is conormal in the preceding subgroup (i.e. X_{i+1} is conormal in X_i for $0 \leq i \leq n-1$) and is normal in G. In other words (X_{i+1}, X_i) is a subcofactor for $0 \leq i \leq n-1$. A refinement of a conormal series is a conormal series which contains all of the subgroups in the original series.

Two conormal series $G = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n = 0$ and $G = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m = 0$ in a group G are said to be *projectively isomorphic* if each subcofactor (X_{i+1}, X_i) projects onto one of the subcofactors (Y_{j+1}, Y_j) such that (Y_{j+1}, Y_j) projects onto (X_{i+1}, X_i).

Dual Butterfly Lemma

Theorem

Given two subcofactors (A, A^+) *and* (B, B^+) *of a group, then*

1. *The projection* (C, C^+) *of* (A, A^+) *in* (B, B^+) *and* (D, D^+) *of* (B, B^+) *project onto each other.*
2. $C^+ \backslash C$ *is isomorphic to* $D^+ \backslash D$.

Dual of the Zassenhaus Refinement Theorem

Theorem

Consider two conormal series

$$G = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n = 0$$

and

$$G = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_m = 0.$$

Then the projections of the Y_j *in all* (X_{i+1}, X_i) *and the projections of the* X_i *in all* (Y_{j+1}, Y_j) *form projectively isomorphic series refining the given series. Moreover every subcofactor* (X, X^+) *of* (X_{i+1}, X_i) *onto which some subcofactor of* (Y_{j+1}, Y_j) *projects is smaller than the projection* (M, M^+) *of* (Y_{j+1}, Y_j) *in* (X_{i+1}, X_i)*. Therefore the refinements are the coarsest projectively isomorphic refinements.*

Dual Jordan-Hölder theorem

Definition

A series

$$G = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n = 0$$

is a chief series if it is a maximal conormal series.

Corollary

Consider two chief series

$$G = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n = 0$$

and

$$G = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_m = 0$$

of the same group G. Then the two chief series are projectively isomorphic.

Example

Consider the context of rings with identity and ring homomorphisms. If we take the subgroups of a ring as the additive subgroups. then:

- Conormal subgroups are subrings
- Normal subgroups are ideals.

Not every subgroup is conormal.

References

References

[1] J. Isbell, Zassenhaus' theorem supercedes the Jordan-Hölder theorem, Advances in Mathematics **31** (1979), 101–103.
[2] A. Goswami and Z. Janelidze, Duality in non-abelian algebra IV. Duality for groups and a universal isomorphism theorem, Advances in Mathematics **349** (2019), 781–812.
[3] F. K. van Niekerk, Biproducts and commutators for noetherian forms, Theory and Applications of Categories **34** (2019), 961–992.
[4] H. Zassenhaus, Zum Satz von Jordan-Hölder-Schreier, Abh. Math. Sem. Univ. Hamburg **10** (1934), 106—108.