Dual representations of ortholattices

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Ortholattices

Definition

An ortholattice is a bounded lattice $(L, \wedge, \vee, 0, 1)$ with a unary operation $'$ such that:

1. $a \wedge a' = 0$ and $a \vee a' = 1$
2. $a'' = a$
3. $a \leqslant b \Rightarrow b' \leqslant a'$

Note: $(a \wedge b)' = a' \vee b'$ and $(a \vee b)' = a' \wedge b'$ can be derived from (ii) and (iii).

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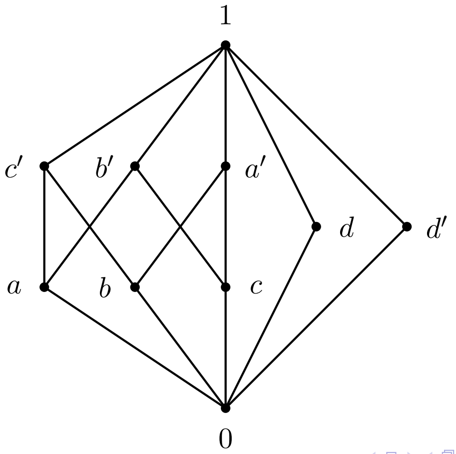
$$x \leqslant y \implies y = x \vee (y \wedge x')$$

is an orthomodular lattice (OML).

MOL $\neq$ OML

Every modular ortholattice is orthomodular, but not conversely.

The smallest orthomodular lattice that is not a modular ortholattice:



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THE LOGIC OF QUANTUM MECHANICS

BY GARRETT BIRKHOFF AND JOHN VON NEUMANN

(Received April 4, 1936)

1. Introduction. One of the aspects of quantum theory which has attracted the most general attention, is the novelty of the logical notions which it presupposes. It asserts that even a complete mathematical description of a physical system $\mathfrak{S}$ does not in general enable one to predict with certainty the result

More about ortholattices

- H an inner product space with $S \subseteq H$

$$S^{\perp} = \{\, x \in H \mid (\forall\, s \in S)(x \perp s) \,\}$$

The subspaces of H form an ortholattice.

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  The subspaces of H form an ortholattice.

- If H is a Hilbert space, then the lattice of subspaces is an orthomodular lattice.
- Recall the distributive laws for lattices:

$$\begin{aligned} &(\mathrm{D1}) \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \\ &(\mathrm{D2}) \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \end{aligned}$$

  If an ortholattice $\mathbb{L} = (L, \wedge, \vee, ', 0, 1)$ is distributive, then it is a Boolean algebra.

Goldblatt's representation

Definition

An orthogonality space $\mathcal{F} = (X, \perp)$ is a set X with a binary relation $\perp \subseteq X^2$ that is irreflexive and symmetric.

$$Y^{\perp} = \{\, x \in X \mid (\forall y \in Y)(x \perp y) \,\}$$

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Proposition (cf. Birkhoff 1967)

Let $\mathcal{F} = (X, \perp)$ be an orthogonality space. Then

$$(\{\, Y \subseteq X \mid Y = Y^{\perp\perp} \,\}, \subseteq)$$

is a complete ortholattice (with orthocomplement $^{\perp}$).

Goldblatt's representation

We recall the following familiar definition:

Definition

Let $\mathbb{K} = (K, \wedge, \vee)$ be a lattice.

$F \subseteq K$ is a filter of $\mathbb{K}$ if

- F is an up-set (i.e. $x \in F$ & $x \leqslant y \implies y \in F$)
- $x, y \in F \implies x \wedge y \in F$

We denote the set of filters of $\mathbb{K}$ by Filt$(\mathbb{K})$.

NB: if $\mathbb{K}$ finite, every filter $F = \uparrow x$ for some $x \in K$.

Goldblatt's representation

Let $\mathbb{L} = (L, \wedge, \vee, ', 0, 1)$ be an ortholattice.

- Filt($\mathbb{L}$) is the set of filters of $\mathbb{L}$
- For $F, G \in$ Filt($\mathbb{L}$)

$$F \perp G \quad \iff \quad a \in F,\ a' \in G$$

- Define $\varphi(a) = \{\, F \in \mathsf{Filt}(\mathbb{L}) \mid a \in F \,\}$
- Let

$$\mathcal{S} = \{\, \varphi(a) \mid a \in L \,\} \cup \{\, (\varphi(a))^{\complement} \mid a \in L \,\}$$

be a subbase for τ, the topology on $\mathcal{X}(\mathbb{L}) = (\mathsf{Filt}(\mathbb{L}), \perp, \tau)$.

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Any ortholattice $\mathbb{L}$ is isomorphic to the clopen $\perp$-closed subsets of $\mathcal{X}(\mathbb{L})$

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Theorem (Goldblatt 1975)

Any ortholattice $\mathbb{L}$ is isomorphic to the clopen $\perp$-closed subsets of $\mathcal{X}(\mathbb{L})$ (via the isomorphism $a \mapsto \varphi(a)$).

A representation via digraphs (Ploščica 1995)

For a lattice $\mathbb{K}$, let $D(\mathbb{K})$ be the set of all maximal disjoint filter-ideal pairs of $\mathbb{K}$.

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For $\langle F, I\rangle$ and $\langle G, J\rangle$ in $D(\mathbb{K})$ define:

$$\langle F, I\rangle E \langle G, J\rangle \iff F \cap J = \emptyset$$

$(D(\mathbb{K}), E)$ is a digraph with loops on each vertex

The dual digraphs of lattices have been characterised as *TiRS digraphs* (C., Gouveia, Haviar 2015).

Let (V, E) be a digraph. Define:

$$E_{\triangleright}^{\complement}(Y) = \{\, x \in X \mid (\forall\, y \in Y)(y, x) \notin E \,\}$$

$$E_{\triangleleft}^{\complement}(Y) = \{\, z \in X \mid (\forall\, y \in Y)(z, y) \notin E \,\}$$

The complete lattice $\mathcal{C}(V, E)$ is given by:

$$\mathcal{C}(V, E) = \{\, Y \subseteq V \mid (E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement})(Y) = Y \,\},$$

ordered by inclusion.

Adding topology to dual digraphs

For $x \in K$, define:

$A_x = \{\langle F, I\rangle \in D(\mathbb{K}) \mid a \in I\}$ and $B_x = \{\langle F, I\rangle \in D(\mathbb{K}) \mid a \in F\}$.

Add a topology with subbase of closed sets:

$$\{A_x \mid x \in K\} \cup \{B_x \mid x \in K\}$$

Unary map on dual digraphs

Let $\mathbf{G} = (V, E)$ be a TiRS digraph and $N : V \to V$. For $xE = \{y \in V \mid xEy\}$ and $Ex = \{y \in V \mid yEx\}$, consider the following properties.

M1 $N(N(x)) = x$

M2 $xE \supseteq yE \implies EN(x) \supseteq EN(y)$

M3 $Ex \supseteq Ey \implies N(x)E \supseteq N(y)E$

O $\forall x \exists y : xE \supseteq yE \quad \& \quad EN(x) \supseteq Ey$

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Proposition

Let $\mathbb{L}$ be an ortholattice with dual digraph $(D(\mathbb{L}), E)$. Define $N_{\mathbb{L}} : D(\mathbb{L}) \to D(\mathbb{L})$ for $\langle F, I \rangle$ by $N(\langle F, I \rangle) = \langle I', F' \rangle$. Then N satisfies the properties above.

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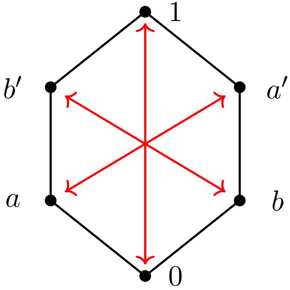
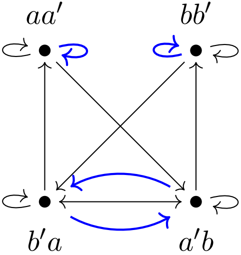
Theorem

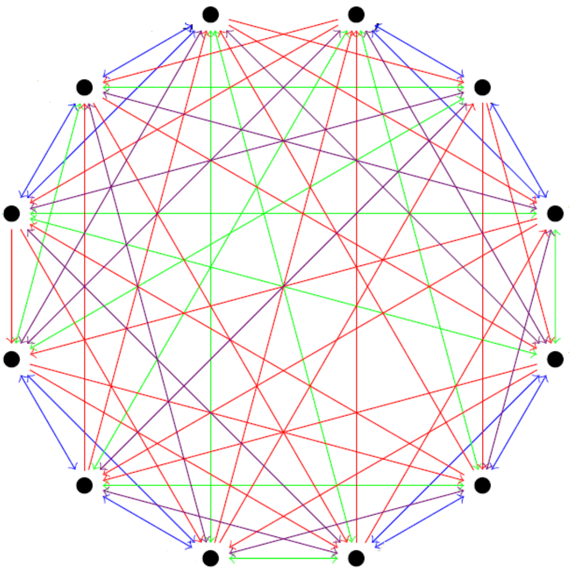
Every ortholattice is isomorphic to the doubly-closed elements of $\mathcal{C}(D(\mathbb{L}), E)$. (The unary map $N_{\mathbb{L}}$ is used to define the $'$.)





The ortholattice $\mathbb{O}_6$ (complement in red) and $(D(\mathbb{O}_6), E, N)$ with its negation in blue.

Dual digraph of MO_2



Further work

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- Simplify topological conditions on dual spaces.
  - Goldblatt does not provide characterisation of dual topological spaces.
  - Our current characterisation of the topological conditions is a direct translation of the L-spaces of Urquhart. Can this be simplified in the TiRS digraph setting? What if we restrict to dual spaces of ortholattices?