When are mixed convergences topological?

Jurie Conradie

University of Cape Town

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University of Stellenbosch
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Joint work with

Michael O'Brien, Vladimir Troitsky and Jan Harm van der Walt.

Lebesque's dominated convergence theorem

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(a) there is a $0 \leq g \in L_1(\Omega, \Sigma, \mu)$ such that $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and $|f(x)| \leq g(x)$ almost everywhere,

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then $\int |f_n - f| \, d\mu \to 0$, i.e. (f_n) converges to f in the L_1-norm.

Topological convergence: nets

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Then (x_α) converges to x (written $x_\alpha \to x$) if for every $N \in \mathcal{N}(x)$, there is an $\alpha_0 \in A$ such that $x_\alpha \in N$ for all $\alpha \geq \alpha_0$.

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A filter $\mathcal{F}$ on a set X is a non-empty collection of non-empty subsets of X such that

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A filter $\mathcal{F}$ converges to x in X (written $\mathcal{F} \to x$) if $\mathcal{F} \supseteq \mathcal{N}(x)$.

Topological convergence: the link between nets and filters

If $(x_\alpha)_{\alpha\in A}$ is a net and $\alpha_0 \in A$, then we call the set $\{x_\alpha : \alpha \geq \alpha_0\}$ a **tail** of the net.

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The set

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Conversely, given a filter $\mathcal{F}$ on X, it is possible to construct a net $(x_\alpha)_{\alpha \in A}$ such that

$$\mathcal{F} = \{F \subseteq X : F \supseteq \{x_\alpha : \alpha \geq \alpha_0\}, \alpha_0 \in A\},$$

and such that if $\mathcal{F} \to x$, $x_\alpha \to x$.

Convergence structures

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Concepts (such as compactness) are defined in terms of convergence rather than open sets.

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(b) $\mathcal{F}, \mathcal{G} \in \lambda(x) \Rightarrow \mathcal{F} \cap \mathcal{G} \in \lambda(x)$ for every $x \in X$;

(c) $\mathcal{F} \in \lambda(x), \mathcal{G} \in \mathtt{Fil}(X)$ and $\mathcal{F} \supseteq \mathcal{F} \Rightarrow \mathcal{G} \in \lambda(x)$.

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We write $\mathcal{F} \in \lambda(x)$ as $\mathcal{F} \to x$ in (X, λ), or simply as $\mathcal{F} \to x$ in X. In this case we say $\mathcal{F}$ ***converges*** to x, or x is the ***limit*** of $\mathcal{F}$.

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(b) if $(x_\alpha), (y_\alpha) \in \eta(x)$, then $(z_\alpha) \in \eta(x)$, where for each α, $z_\alpha \in \{x_\alpha, y_\alpha\}$.

(c) if $(x_\alpha) \in \eta(x)$, then $(y_\beta) \in \eta(x)$ for every quasi-subnet (y_β) of (x_α);

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The pair (X, η), or $(X, \to)$, is called a **net convergence space**.

Equivalence of net and filter convergence structures

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In a specific context, one can use filter or net convergence structures, whichever is more convenient.

In what follows, "convergence" will stand for either form of convergence.

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If a convergence structure on X can be derived from a topology on X, we call the convergence **topological**.

Examples of convergence structures (2)

If (Ω, Σ, μ) is a measure space and M the set of all measurable real-valued functions on X. A sequence (f_n) in M **converges almost everywhere** to $f \in M$ iff $f_n(x) \to f(x)$ for every $x \in \Omega \setminus \Omega_0$, where $\mu(\Omega_0) = 0$.

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This type of convergence is, in general, not topological.

Examples of convergence structures (3)

Let K be a topological space and denote by $C(K)$ the space of all continuous real-valued functions on K. Define a net convergence structure on $C(K)$ by $f_\alpha \xrightarrow{c} f$ iff whenever $x_\beta \to x$ in the topology of K we have $f_\alpha(x_\beta) \to f(x)$. The convergence $\xrightarrow{c}$ is known as **continuous convergence.**

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Convergence vector spaces

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A **convergence vector space** is a vector space equipped with a convergence structure for which addition and scalar multiplication are continuous.

Riesz spaces

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An example: If (Ω, Σ, μ) is a measure space, the vector space $L_0(\Omega, \Sigma, \mu)$ of all (equivalence classes modulo almost everywhere equality) of real-valued measurable functions becomes a vector lattice when equipped with the partial order $f \leq g$ iff $f(x) \leq g(x)$ for almost every $x \in \Omega$.

Solid sets

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A **locally solid** convergence vector lattice is a vector lattice equipped with a vector space convergence structure such that if $x_\alpha \to 0$ and $|y_\alpha| \leq |x_\alpha|$ for every α implies $y_\alpha \to 0$.

Atomic Riesz spaces

An element $a > 0$ in a Riesz space E is an atom if the solid linear subspace generated by a,

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The Riesz space E is atomic (or discrete) if it contains a complete disjoint system $\{x_i : i \in I\}$ of atoms, i.e. $x_i \wedge x_j = 0$ for all $i, j \in I, i \neq j$ and $x_i \wedge x = 0$ for all $i \in I$ implies $x = 0$.

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E is atomic iff it is lattice isomorphic to a order dense sublattice of a vector lattice of the form $\mathbb{R}^A$.

Typical examples of atomic Riesz spaces are the spaces ℓ_p and c_0.

Order convergence

A net $(x_\alpha)_{\alpha \in A}$ in a Riesz space E **converges in order** to x in E (denoted $x_\alpha \xrightarrow{o} x$) if there is a net $(y_\beta)_{\beta \in B}$ in E^+ such that $y_\beta \downarrow 0$ and for every $\beta \in B$ there is an $\alpha_\beta \in A$ such that $|x_\alpha - x| \le y_\beta$ for every $\alpha \ge \alpha_\beta$.

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In the case $E = L_1$, $f_n \xrightarrow{o} 0$ iff $f_n \to 0$ almost everywhere and some tail of (f_n) is order-bounded.

Unbounded order convergence

A net (x_α) in a Riesz space E converges unboundedly in order to x in E (denoted $x_\alpha \xrightarrow{uo} x$) if for every $0 < u \in E$, $|x_\alpha - x| \wedge u \xrightarrow{o} 0$.

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For the sequence (e_n) in c_0, we have $e_n \xrightarrow{uo} 0$, but (e_n) does not converge in order to 0.

Vector bornologies

A family $\mathcal{B}$ of subsets of a vector space E is a *vector bornology* on E if

(a) $\bigcup\{B : B \in \mathcal{B}\} = E$;

(b) $B \in \mathcal{B}, C \subseteq B \Rightarrow C \in \mathcal{B}$;

(c) $B_1, B_2 \in \mathcal{B} \Rightarrow B_1 + B_2 \in \mathcal{B}$;

(d) $B \in \mathcal{B}, 0 \neq \lambda \in \mathbb{R} \Rightarrow \lambda B \in \mathcal{B}$;

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(e) $\mathcal{B}$ is closed under the formation of balanced hulls.

A subclass $\mathcal{B}_0$ of $\mathcal{B}$ is a basis for $\mathcal{B}$ if for every $B \in \mathcal{B}$, there is a $B_0 \in \mathcal{B}_0$ such that $B \subseteq B_0$.

Mixed convergence (nets)

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$(E, \rightarrow_{\mathcal{A}})$ is also a convergence vector space.

If E is a vector lattice and $\rightarrow$ a locally solid vector convergence on E and $\mathcal{A}$ a solid bornology on E, then $\rightarrow_{\mathcal{A}}$ is a locally solid convergence.

Mixed convergence (filters)

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There is an equivalent definition for mixed convergences in terms of filters:

If (E, λ) is a filter convergence space and $\mathcal{A}$ a vector bornology on E. The mixed (filter) convergence $\lambda_{\mathcal{A}}$ on E is defined by

$$\mathcal{F} \in \lambda_{\mathcal{A}}(0) \Leftrightarrow \mathcal{F} \in \lambda(0) \text{ and } \mathcal{F} \cap \mathcal{A} \neq \emptyset.$$

Examples of mixed convergences(1)

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The mixed convergence $\xrightarrow{uo}_{\mathcal{A}}$ is order convergence ($\xrightarrow{o}$).

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Let $\xrightarrow{w*}$ denote convergence with respect to the weak*-topology on $\mathcal{L}(E)$ (i.e. $f_\alpha \xrightarrow{w*} 0 \Leftrightarrow f_\alpha(x) \to 0$ for every $x \in E$).

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Let $\xrightarrow{w*}$ denote convergence with respect to the weak*-topology on $\mathcal{L}(E)$ (i.e. $f_\alpha \xrightarrow{w*} 0 \Leftrightarrow f_\alpha(x) \to 0$ for every $x \in E$).

The mixed convergence $\xrightarrow{w*}_{\mathcal{A}}$ is continuous convergence ($\xrightarrow{c}$) on $\mathcal{L}(E)$, a vector subspace of $C(E)$.

Mixed convergences as final convergences

If $(E, \mathcal{A}, \rightarrow)$ is a mixed convergence space, then the mixed convergence $\rightarrow_{\mathcal{A}}$ on E is the finest convergence on E for which the embedding

$$e_A : (A, \rightarrow) \hookrightarrow E$$

is continuous for each $A \in \mathcal{A}$.

Mixed convergences as final convergences

If $(E, \mathcal{A}, \to)$ is a mixed convergence space, then the mixed convergence $\to_{\mathcal{A}}$ on E is the finest convergence on E for which the embedding

$$e_A : (A, \to) \hookrightarrow E$$

is continuous for each $A \in \mathcal{A}$.

Equivalently, the convergence $\to_{\mathcal{A}}$ is the finest convergence on E which agrees with the convergence $\to$ on the sets in $\mathcal{A}$.

When is a vector convergence topological?

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For a convergence space (E, λ), we define the neighbourhood filter of $x \in E$ by

$$\mathcal{N}_\lambda(x) = \cap\{\mathcal{F} \in \mathtt{Fil}(E) : \mathcal{F} \in \lambda(x)\},$$

i.e. $\mathcal{N}_\lambda(x)$ is the intersection of all filters λ-converging to x.

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A vector convergence space (E,λ) is topological iff $\mathcal{N}_\lambda(0) \in \lambda(0)$ (i.e. $\mathcal{N}_\lambda(0)$ converges to 0).

When is a mixed convergence topological?

$$
\begin{aligned}
\lambda_{\mathcal{A}} \text{ is topological} \quad &\Leftrightarrow \quad \mathcal{N}_{\lambda_{\mathcal{A}}}(0) \in \lambda_{\mathcal{A}}(0) \\
&\Leftrightarrow \quad \mathcal{N}_{\lambda_{\mathcal{A}}}(0) \in \lambda(0) \text{ and } \mathcal{N}_{\lambda_{\mathcal{A}}}(0) \cap \mathcal{A} \neq \emptyset
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Since $\mathcal{N}_{\lambda_{\mathcal{A}}}(0) \subseteq \mathcal{N}_{\lambda}(0)$, it follows that a necessary condition for $\lambda_{\mathcal{A}}$ to be topological is that λ be topological.

Topological in, topological out?

If we start with a topological vector convergence $\rightarrow$ on a vector space E, a vector bornology $\mathcal{A}$ on E and look at a corresponding mixed convergence $\rightarrow_{\mathcal{A}}$, will the result be a topological convergence as well?

Not necessarily

Let E be Riesz space and consider unbounded order convergence $\xrightarrow{uo}$ on E. As bornology $\mathcal{A}$ we take the set of order-bounded sets in E.

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However, $\xrightarrow{uo}_{\mathcal{A}} = \xrightarrow{o}$ is topological iff E is finite-dimensional.

Thus the mixed convergence $\xrightarrow{uo}_{\mathcal{A}}$ is topological iff E is both atomic and finite-dimensional (and so lattice isomorphic to $\mathbb{R}^n$, for some n).

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The mixed convergence is topological iff E is locally compact.

This is the case iff E is finite-dimensional, and in this case $\mathcal{L}(E)$ is finite-dimensional as well.

Lebesque's dominated convergence theorem

Let (Ω, Σ, μ) be a measure space and $f_1, f_2, f_3, \ldots, f \in L_1(\Omega, \Sigma, \mu)$. Then if

(a) there is a $0 \leq g \in L_1(\Omega, \Sigma, \mu)$ such that $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and $|f(x)| \leq g(x)$ almost everywhere,

(b) and either
   - (f_n) convergences almost everywhere to f, or
   - (f_n) convergences to f in measure on sets of finite measure,

then $\int |f_n - f| \, d\mu \to 0$, i.e. (f_n) converges to f in the L_1-norm.