On graded algebras

Simon Thulani Chili

University of Cape Town

06 December 2022

S-graded K-algebras

Definition

Given a semigroup S and a unital commutative ring K, by an S-graded K-algebra we mean an S-indexed family $A = (A_s)_{s \in S}$ of K-modules equipped with a family of K-bilinear multiplications

$$(A_s \times A_t \to A_{st})_{s,t \in S}$$

all written as $(x, y) \mapsto xy$ and satisfying the associative condition $x(yz) = (xy)z$ for all $s, t, u \in S$ and $x \in A_s$, $y \in A_t$, $z \in A_u$. A morphism $f : A \to B$ of S-graded K-algebras is a family $f = (f_s)_{s \in S}$ of K-module homomorphisms $f_s : A_s \to B_s$ with $f_s(x) f_t(y) = f_{st}(xy)$ for all $s, t \in S$ and $x \in A_s$, $y \in A_t$.

We denote the category of S-graded K-algebras by $\mathbf{Alg}(K, S)$.

Smith and Huq commutators

Definition

Let $A = (A_s)_{s \in S}$ be an S-graded K-algebra, a congruence $\alpha = (\alpha_s)_{s \in S}$ on A is an S-indexed family of congruences with α_s a congruence on A_s such that for each $s, t \in S$, $(a, b) \in \alpha_s$ and $(c, d) \in \alpha_t$ we have $(ac, bd) \in \alpha_{st}$.

Definition

Let $A = (A_s)_{s \in S}$ be an S-graded K-algebra, an ideal $I = (I_s)_{s \in S}$ on A is an S-indexed family of K-submodules where I_s is a K-submodule of A_s such that for each $s, t \in S$ we have $xa \in I_{st}$ and $by \in I_{st}$ whenever $x \in I_s$, $a \in A_t$, $b \in A_s$ and $y \in I_t$.

Proposition

Given an S-graded K-algebra $A = (A_s)_{s \in S}$, the maps $\lambda : Con(A) \to I(A)$ and $\gamma : I(A) \to Con(A)$ defined, respectively, for each $s \in S$ by $\lambda(\alpha)_s = [0]_{\alpha_s}$ and $(a, b) \in \gamma(I)_s \iff a - b \in I_s$ are order preserving bijections (inverse to each other).

Smith and Huq commutators

Definition

Given congruences $\alpha = (\alpha_s)_{s \in S}$ and $\beta = (\beta_s)_{s \in S}$ on an S-graded K-algebra $A = (A_s)_{s \in S}$, the Smith commutator $[\alpha, \beta]_S$ is the smallest congruence $\theta = (\theta_s)_{s \in S}$ on A such that the family of functions

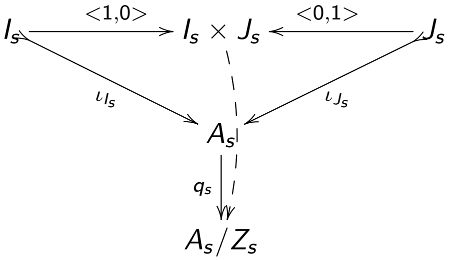
$$\left(A_s^* = \{(x, y, z) \in A_s^3 | (x, y) \in \alpha_s \text{ and } (y, z) \in \beta_s\} \longrightarrow A_s/\theta_s \right)_{s \in S}$$

defined (for each $s \in S$) by sending triple (x, y, z) to class $[p(x, y, z) = x - y + z]_{\theta_s}$ defines a morphism of S-graded K-algebras.

Smith and Huq commutators

Definition

Given two ideals (=normal S-graded K-subalgebras) $I = (I_s)_{s \in S}$ and $J = (J_s)_{s \in S}$ of S-graded K-algebra $A = (A_s)_{s \in S}$, the Huq commutator $[I, J]_Q$ of ideals I and J of A is the smallest ideal Z of A such that there exits K-linear maps $I_s \times J_s \to A_s/Z_s$ forming a morphism of S-graded K-algebras and such that for each $s \in S$ the following diagram.



commutes.

Smith and Huq commutators

Proposition

The category $Alg(K,S)$ of S-graded K-algebras is strongly protomodular.

- Given S-graded K-algebras $E = (E_s)_{s \in S}$, $B = (B_s)_{s \in S}$ and a morphism $p = (p_s : E_s \to B_s)_{s \in S}$. We have that the change of base functor $p^* : Pt_{\mathbf{Alg}(K,S)}(B) \to Pt_{\mathbf{Alg}(K,S)}(E)$ is normal.
- This functor preserves finite limits, reflects isomorphisms and normal monomorphisms.

Smith and Huq commutators

Proposition

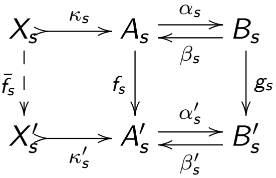
Given two ideals $I = (I_s)_{s\in S}$ and $J = (J_s)_{s\in S}$ of S-graded K-algebra $A = (A_s)_{s\in S}$, the Huq commutator $[I, J]_Q$ is the ideal on A with $([I, J]_Q)_u = \sum_{u=st}(I_sJ_t + J_sI_t)$

- Let (for each $s \in S$) $\varphi_s : I_s \times J_s \to A_s/Z_s$ be a K-linear map be defined by $\varphi_s(x, y) = x + y + Z_s$.
- To require the family $\varphi = (\varphi_s)_{s\in S}$ to be a morphism of S-graded K-algebras is to require that for each $x \in I_s, x' \in I_t$ and $y \in J_s, y' \in J_t$,
$xx' + yy' + Z_{st} = (x + y + Z_s)(x' + y' + Z_t)$, that is,
$xx' + yy' + Z_{st} = xx' + xy' + yx' + yy' + Z_{st}$, and equivalently
$xy' + yx' + Z_{st} = Z_{st}$. We therefore have that the Huq commutator $[I, J]_Q$ is given by $([I, J]_Q)_u = \sum_{u=st}(I_sJ_t + J_sI_t)$.

Actions and split extensions

Definition

A split extension of S-graded K-algebras is a system $(A, B, X, \alpha, \beta, \kappa)$ where $A = (A_s)_{s \in S}$, $B = (B_s)_{s \in S}$ and $X = (X_s)_{s \in S}$ are S-graded K-algebras, $\alpha = (\alpha_s : A_s \to B_s)_{s \in S}$ a split epimorphism with specified splitting $\beta = (\beta_s : B_s \to A_s)_{s \in S}$ and $\kappa = (\kappa_s : X_s \to A_s)_{s \in S}$ a fixed kernel of α.
A morphism $(f, g) : (A, B, X, \alpha, \beta, \kappa) \to (A', B', X', \alpha', \beta', \kappa')$ of split extensions is a pair $(f, g) = (f_s : A_s \to A'_s, g_s : B_s \to B'_s)_{s \in S}$ of morphisms of S-graded K-algebras such that for each $s \in S$ the following diagram



commutes.

Actions and split extensions

Definition

An action of S-graded K-algebras is a system (B, X, l, r) where $B = (B_s)_{s\in S}$ and $X = (X_s)_{s\in S}$ are S-graded K-algebras, $l = (l_{s,t} : B_s \times X_t \to X_{st})_{s,t\in S}$ and $r = (r_{s,t} : X_s \times B_t \to X_{st})_{s,t\in S}$ are a family of maps satisfying the following conditions for all $s, t, u \in S$, $b \in B_s$, $b' \in B_u$ and $x \in X_t$, $x' \in X_u$:

(a) each $l_{s,t}$ and each $r_{s,t}$ are bilinear;

(b) $l_{s,tu}(b, xx') = l_{s,t}(b, x)x'$ and $x' r_{t,s}(x, b) = r_{ut,s}(x'x, b)$;

(c) $l_{s,tu}(b, r_{t,u}(x, b')) = r_{st,u}(l_{s,t}(b, x), b')$ and $x' l_{s,t}(b, x) = r_{u,s}(x', b)x$;

(d) $l_{s,ut}(b, l_{u,t}(b', x)) = l_{su,t}(bb', x)$ and $r_{t,su}(x, bb') = r_{ts,u}(r_{t,s}(x, b), b')$.

A morphism $(f, g) : (B, X, l, r) \to (B', X', l', r')$ of actions is a pair $(f, g) = (f_s : B_s \to B'_s, g_s : X_s \to X'_s)_{s\in S}$ of morphisms of S-graded K-algebras such that $l_{s,t}(f_s(b), g_t(x)) = g_{st}(l_{s,t}(b, x))$ and $r_{t,s}(g_t(x), f_s(b)) = g_{ts}(r_{t,s}(x, b))$.

Actions and split extensions

Given action (B, X, l, r) we associate to it a split extension $(B \ltimes X, B, X, \pi_1, \iota_1, \iota_2)$ where (for each $s, t \in S$)

- $(B \ltimes X)_s = B_s \times X_s$
- with multiplications $(B_s \times X_s) \times (B_t \times X_t) \to B_{st} \times X_{st}$ defined by $(b, x)(b', x') = (bb', l_{s,t}(b, x') + r_{s,t}(x, b') + xx')$.
- and π_1, ι_1, ι_2 defined by $(\pi_1)_s(b, x) = b$, $(\iota_1)_s(b) = (b, 0)$ and $(\iota_2)_s(x) = (0, x)$, respectively.

Conversely, to a split extension $(A, B, X, \alpha, \beta, \kappa)$ we associate to it an action $(B, Ker(\alpha), l, r)$, where

- $l = (l_{s,t} : B_s \times Ker(\alpha_t) \to Ker(\alpha_{st}))_{s,t \in S}$ and
- $r = (r_{s,t} : Ker(\alpha_s) \times B_t \to Ker(\alpha_{st}))_{s,t \in S}$

are defined by $l_{s,t}(b, x) = \beta_s(b)x$ and $r_{s,t}(x, b) = x\beta_t(b)$, respectively.
Resulting in equivalence of categories $\mathbf{SplExt}(K, S) \sim \mathbf{Act}(K, S)$.

Internal categories and crossed modules

An internal category in $\mathbf{Alg}(K,S)$ is a system (C_0, C_1, d, c, e, m)=

$$\left((C_1)_s \times_{(d_s,c_s)} (C_1)_s \xrightarrow{m_s} (C_1)_s \underset{c_s}{\overset{d_s}{\rightrightarrows}} (C_0)_s, \quad (C_0)_s \xrightarrow{e_s} (C_1)_s \right)_{s\in S}$$

satisfying the usual conditions required for a category.

Proposition

A system (C_0, C_1, d, c, e, m) in $\boldsymbol{Cat}(K,S)$ the category of internal categories in the category $\boldsymbol{Alg}(K,S)$ of S-graded K-algebras is completely determined by the reflexive graph (C_0, C_1, d, c, e) in $\boldsymbol{Alg}(K,S)$ with vanishing commutator condition $[Ker(d), Ker(c)]_Q = 0$.

Internal categories and crossed modules

Given reflexive graph (C_0, C_1, d, c, e) in $\mathbf{Alg}(K, S)$ with vanishing commutator condition $[Ker(d), Ker(c)]_Q = 0$.
Define $m = (m_s : (C_1)_s \times_{(d_s,c_s)} (C_1)_s \to (C_1)_s)_{s\in S}$ by
$m_s(g, f) = g - 1_{d_s(g)=c_s(f)} + f$.
For diagrams $x \xrightarrow{f} y \xrightarrow{g} z$, in $(C_1)_s$ and $x' \xrightarrow{f'} y' \xrightarrow{g'} z'$ in $(C_1)_t$ we have $(g - 1_y + f)(g' - 1_{y'} + f') =$
$gg' - 1_{yy'} + ff' + (g - 1_y)(f' - 1_{y'}) + (f - 1_y)(g' - 1_{y'})$

- Note that $[(g-1_y)(f'-1_{y'})+(f-1_y)(g'-1_{y'})] \in [Ker(d), Ker(c)]_Q = 0$
- And if maps $m_s(g, f) = g - 1_y + f$ form a morphism of S-graded K-algebra we will have $(g - 1_y)(f' - 1_{y'}) + (f - 1_y)(g' - 1_{y'}) = 0$, in particular, if $y = y' = 0$, we get $gf' + fg' = 0$ and hence $[Ker(d), Ker(c)]_Q = 0$.

Internal categories and crossed modules

Definition

A crossed module of S-graded K-algebras is a system (B, X, l, r, δ) where (B, X, l, r) is an object in $\mathbf{Act}(K, S)$ and $\delta = (\delta_s : X_s \to B_s)_{s \in S}$ is a morphism of S-graded K-algebras satiesfying the following conditions for all $s, t \in S$, $b \in B_s$, $x \in X_t$ and $x' \in X_u$:

(a) $\delta_{st}(l_{s,t}(b, x)) = b\delta_t(x)$ and $\delta_{ts}(r_{t,s}(x, b)) = \delta_t(x)b$;

(b) $l_{t,u}(\delta_t(x), x') = xx' = r_{t,u}(x, \delta_u(x'))$.

A morphism $(f, g) : (B, X, l, r, \delta) \to (B', X', l', r', \delta')$ of crossed modules is a morphism $(f, g) : (B, X, l, r) \to (B', X', l', r')$ of actions of S-graded K-algebras such that $f_s\delta_s = \delta'_s g_s$.

Internal categories and crossed modules

Given internal category (C_0, C_1, d, c, e, m) we associate to it crossed module $(C_0, Ker(d), r, l, \delta)$ where

- $r = (r_{s,t} : (C_0)_s \times Ker(d_t) \to Ker(d_{st}))_{s,t \in S}$
- $l = (l_{s,t} : Ker(d_s) \times (C_0)_t \to Ker(d_{st}))_{s,t \in S}$
- $\delta = (\delta_s : Ker(d_s) \to (C_0)_s)_{s \in S}$

are defined by $r_{s,t}(a, x) = e_s(a)x$, $l_{s,t}(x, a) = xe_t(a)$ and $\delta_s(x) = c_s(x)$.

Internal categories and crossed modules

Conversely, to a crossed module (B, X, r, l, δ) we associate to it an internal category $(B, B \ltimes X, d, c, e, m)$ where (for each $s \in S$)

- $(B \ltimes X)_s = B_s \times X_s$
- $d = (d_s : B_s \times X_s \to B_s)_{s \in S}$
- $c = (c_s : B_s \times X_s \to B_s)_{s \in S}$
- $e = (e_s : B_s \to B_s \times X_s)_{s \in S}$
- $m = (m_s : (B_s \times X_s) \times_{(d_s, c_s)} (B_s \times X_s) \to B_s \times X_s)_{s \in S}$

are defined by $d_s(b, x) = b$, $c_s(b, x) = \delta_s(x) + b$, $e_s(b) = (b, 0)$ and $m_s((b', x'), (b, x)) = (b, x' + x)$ (with $b' = \delta_s(x) + b$).
Resulting in an equivalence of categories
$\mathbf{Cat}(K, S) \sim \mathbf{XMod}(K, S)$.

Thank you!