A Lumer–Phillips type generation theorem for bi-continuous semigroups

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Outline

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and the corresponding abstract Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = \mathcal{A}u(t,x), & t \ge 0, \\ u(0,x) = f(x) \in \mathrm{C_b}(\mathbb{R}^n). \end{cases}$$

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- $(T(t))_{t \geq 0}$ is not strongly continuous on $\mathrm{C_b}(\mathbb{R}^n)$

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$$\|x\| = \sup_{\substack{\varphi \in (X,\tau)' \\ \|\varphi\| \leq 1}} |\varphi(x)|$$

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Examples: $\mathrm{C_b}(\mathbb{R})$ with compact-open topology, $\mathscr{L}(E)$ with strong operator topology, X' with weak*-topology

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(iii) $(T(t))_{t\geq 0}$ is locally bi-equicontinuous, i.e., for every $\|\cdot\|$-bounded sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \xrightarrow{\tau} 0$ we have $T(t)x_n \xrightarrow{\tau} 0$ uniformly on bounded intervals.

Examples

- Left-translation semigroup on $X = \mathrm{C_b}(\mathbb{R})$ defined by $(T(t)f)(x) := f(x+t)$, $t \geq 0$, $f \in \mathrm{C_b}(\mathbb{R})$, $x \in \mathbb{R}$.

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$$Ax := \tau\lim_{t\to 0} \frac{T(t)x - x}{t}$$

$$\mathrm{D}(A) := \left\{ x \in X : \ \exists \ \tau\lim_{t\to 0} \frac{T(t)x - x}{t}, \ \sup_{t\in(0,1]} \frac{\|T(t)x - x\|}{t} < \infty \right\}$$

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The classical Lumer–Phillips theorem

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A linear operator $(A, \mathrm{D}(A))$ on a Banach space X is called *dissipative* if

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for all $\lambda > 0$ and $x \in \mathrm{D}(A)$.

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Preperations for the bi-continuous case – I

Theorem (F. Kühnemund)

Let $(X, \|\cdot\|, \tau)$ *be a bi-admissible space, and let* $(A, \mathrm{D}(A))$ *be a linear operator on the Banach space* X*. The following are equivalent:*

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a. *The operator $(A, \mathrm{D}(A))$ is the generator of a bi-continuous contraction semigroup $(T(t))_{t\geq 0}$.*
b. *The operator $(A, \mathrm{D}(A))$ satisfies $(0,\infty) \subseteq \rho(A)$ and*

$$\|R(\lambda, A)^n\| \leq \frac{1}{\lambda^n} \tag{1}$$

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is bi-equicontinuous, meaning that for each norm bounded τ-null sequence $(x_m)_{m\in\mathbb{N}}$ one has $\lambda^n R(\lambda, A)^n x_m \to 0$ in τ uniformly for $n \in \mathbb{N}$ and $\lambda > 0$ as $n \to \infty$.

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Definition

Let $(A, \mathrm{D}(A))$ be an operator on a bi-admissible space $(X, \|\cdot\|, \tau)$. We call this operator *bi-dissipative* if there exists a fundamental system of seminorms Γ generating the topology τ such that

$$p((\lambda - A)x) \geq \lambda p(x), \quad \lambda > 0,\ p \in \Gamma,\ x \in \mathrm{D}(A) \tag{3}$$

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$$\|x\| = \sup_{p \in \Gamma} p(x), \quad x \in X. \tag{4}$$

Preparations for the bi-continuous case – III

Proposition

Let $(A, \mathrm{D}(A))$ *be a bi-dissipative operator on a bi-admissible space* $(X, \|\cdot\|, \tau)$*. Then the following assertions are true:*

1. $\lambda - A$ *is injective for all* $\lambda > 0$*. Moreover, one has that*

$$p(R(\lambda, A)x) \leq \frac{1}{\lambda} p(x), \tag{5}$$

   for all $\lambda > 0$, $p \in \mathscr{P}_\tau$ *and* $x \in \mathrm{Ran}(\lambda - A)$*.*

2. $\lambda - A$ *is surjective for some* $\lambda > 0$ *if and only if it is surjective for all* $\lambda > 0$*. In addition, one has that* $(0, \infty) \subseteq \rho(A)$*.*

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The resolvent of $(A, \mathrm{D}(A))$ can be determined explicitly by

$$(R(\lambda, A)f)(x) = \int_0^x \mathrm{e}^{\lambda(t-x)} f(t) \, \mathrm{d}t = \mathrm{e}^{-\lambda x} \int_0^x \mathrm{e}^{\lambda t} f(t) \, \mathrm{d}t, \quad \lambda > 0,$$

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showing that $\lambda - A$ is surjective. Consider the fundamental system of seminorms $(p_n)_{n\in\mathbb{N}}$ with $p_n(f) := \sup_{x\in[0,n]} |f(x)|$. Let $n \in \mathbb{N}$ be arbitrary and observe that for $x \in [0, n]$ one has

$$\lambda \left|(R(\lambda, A)f)(x)\right| \leq \lambda \mathrm{e}^{\lambda x} \int_0^x \mathrm{e}^{\lambda s} \left|f(s)\right| \, \mathrm{d}s \leq \lambda \mathrm{e}^{\lambda x} \int_0^x \mathrm{e}^{\lambda s} p_n(f) \, \mathrm{d}s = (1 - \mathrm{e}^{-\lambda x}) p_n(f).$$

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Hence, $(A, \mathrm{D}(A))$ is a bi-dissipative operator.

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Closeable operators - Banach space vs. locally convex space

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Let $(A, \mathrm{D}(A))$ be a linear operator on a Banach space X. We call the operator *closeable* if for every sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{D}(A)$ with $x_n \to 0$ and $Ax_n \to y$ one has $y = 0$.

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Let $(A, \mathrm{D}(A))$ be a linear operator on a locally convex space (X, τ)

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Let $(A, \mathrm{D}(A))$ be a linear operator on a locally convex space (X, τ). We call the operator *τ-closeable* if for every net $(x_\alpha)_{\alpha\in A}$ in $\mathrm{D}(A)$ with $x_\alpha \to 0$ and $Ax_\alpha \to y$ one has $y = 0$.

The mixed topology

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$$\widetilde{\Gamma} := \left\{ \widetilde{p}_{(a_n),(p_n)} : \ (p_n)_{n\in\mathbb{N}} \subseteq \Gamma, \ (a_n)_{n\in\mathbb{N}} \in \mathrm{c}_0, \ a_n \geq 0 \right\}$$

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where

$$\widetilde{p}_{(a_n),(p_n)}(x) := \sup_{n\in\mathbb{N}} a_n p_n(x), \quad x \in X,$$

Remark

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1. $\tau \subseteq \gamma \subseteq \|\cdot\|$
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Remark

1. $\tau \subseteq \gamma \subseteq \|\cdot\|$
2. *A sequence converges with respect to the mixed topology if and only if it is $\|\cdot\|$-bounded and τ-convergent.*
3. *The classes of bi-continuous semigroups and $\gamma(\tau, \|\cdot\|)$-strongly continuous and locally sequentially $\gamma(\tau, \|\cdot\|)$-equicontinuous semigroups coincide*

A second Lumer–Phillips result

Theorem

Let $(A, \mathrm{D}(A))$ *be a bi-dissipative and bi-densely defined operator on a bi-admissible space* $(X, \|\cdot\|, \tau)$

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Let $(A, \mathrm{D}(A))$ *be a bi-dissipative and bi-densely defined operator on a bi-admissible space* $(X, \|\cdot\|, \tau)$*. Assume that* X *is complete with respect to the mixed topology* γ

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Theorem

Let $(A, \mathrm{D}(A))$ be a bi-dissipative and bi-densely defined operator on a bi-admissible space $(X, \|\cdot\|, \tau)$. Assume that X is complete with respect to the mixed topology γ. If $\mathrm{Ran}(\lambda - A)$ is bi-dense in X for some $\lambda > 0$, then the γ-closure $(\overline{A}, \mathrm{D}(\overline{A}))$ of $(A, \mathrm{D}(A))$ generates a bi-continuous contraction semigroup on X.

Remark

1. *Equip the space $\mathrm{C_b}(\mathbb{R})$ of bounded continuous functions on the real line with the sup-norm $\|\cdot\|_\infty$ and the compact-open topology τ_{co}. Then $\mathrm{C_b}(\mathbb{R})$ is complete with respect to the associated mixed topology $\gamma(\tau_{\mathrm{co}}, \|\cdot\|_\infty)$.*

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2. *The space $\mathscr{L}(E)$ of bounded linear operators on a Banach space E together with the operator norm $\|\cdot\|_{\mathscr{L}(E)}$ and the strong operator topology τ_{sot} is also complete with respect to the mixed topology $\gamma(\tau_{\mathrm{sot}}, \|\cdot\|_{\mathscr{L}(E)})$.*

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Theorem

Let $(A, \mathrm{D}(A))$ *be a bi-dissipative and bi-densely defined operator on a bi-admissible space* $(X, \|\cdot\|, \tau)$. *Assume that* X *is complete with respect to the mixed topology* γ. *If* $\mathrm{Ran}(\lambda - A)$ *is bi-dense in* X *for some* $\lambda > 0$, *then the* γ*-closure* $(\overline{A}, \mathrm{D}(\overline{A}))$ *of* $(A, \mathrm{D}(A))$ *generates a bi-continuous contraction semigroup on* X.

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1. *Equip the space* $\mathrm{C_b}(\mathbb{R})$ *of bounded continuous functions on the real line with the sup-norm* $\|\cdot\|_\infty$ *and the compact-open topology* τ_{co}. *Then* $\mathrm{C_b}(\mathbb{R})$ *is complete with respect to the associated mixed topology* $\gamma(\tau_{\mathrm{co}}, \|\cdot\|_\infty)$.
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Thank you for listening!

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