

A Parallel Hybrid Bregman Subgradient Extragradient Method for a System of Pseudomonotone Equilibrium and Fixed Point Problems

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Table of Contents

- 1 Introduction
- 2 Literature Review
- 3 Preliminaries
- 4 Main Results
- 5 Applications



We consider the Equilibrium Problem (denoted by EP) in the framework of real reflexive Banach space. Let E be the real reflexive Banach space and consider its subset C , which is nonempty, closed and convex. Let the mapping $g : C \times C \rightarrow \mathbb{R}$ be a bifunction. The task of the EP with respect to g is to find a point $x^* \in C$ such that

$$g(x^*, y) \geq 0, \quad \forall \quad y \in C. \quad (1)$$

We denote the set of solutions of (1) by $EP(g)$.



Why the EP?

In the literature, it is well-known that numerous fascinating and complicated problems in nonlinear analysis, like the

- Fixed Point problem
- Nash Equilibria
- Optimization problem (saddle points)
- Variational Inequality
- Complementarity problems



Fixed Point problem

let $T : C \rightarrow C$ be an operator, a point $x \in C$ is called a fixed point of T if $x = Tx$. We denote the set of fixed point of T by $F(T)$.

It has numerous applications both in pure and applied science :

- Control Theory
- Optimization
- Game Theory
- Differential equations
- Economics

Different techniques have been presented for assessing and estimating the fixed points of nonexpansive and quasi-nonexpansive mappings, and for multi-valued Bregman nonexpansive mappings



The Common Solution of the EP and the Fixed Point

Consider the following problem

$$\text{Find } x^* \in C \text{ such that } x^* \in EP(g) \cap F(T), \quad (2)$$

Problem (2) has become an important area of research due to its possible applications in applied science. This happens mainly in image processing, network distribution, signal processing, etc,



Tada and Takahashi [1] presented the following hybrid technique for finding the said element x^* of the set of solutions of monotone EP and the set of fixed points a nonexpansive mapping T in the framework of Hilbert spaces;

$$\left\{ \begin{array}{l} \pi_0 \in C_0 = Q_0 = C, \\ z_n \in C \text{ such that } g(z_n, y) + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ \beta_n = \alpha_n x_n + (1 - \alpha_n) T_{z_n}, \\ C_n = \{v \in C : \|w_n - v\| \leq \|x_n - v\|\}, \\ Q_n = \{v \in C : \langle \pi_0 - x_n, v - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} \pi_0 \end{array} \right.$$



Anh [3] proposed the following hybrid extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping T and the set of solutions of EP involving a pseudomonotone bifunction g in Hilbert spaces:

$$\left\{ \begin{array}{l} \pi_0 \in C_0 = Q_0 = C, \\ y_n = \operatorname{argmin} \left\{ \sigma_n g(\rho_n, y) + \frac{1}{2} \|\rho_n - y\|^2 : y \in C \right\} \\ t_n = \operatorname{argmin} \left\{ \sigma_n g(\rho_n, y) + \frac{1}{2} \|y_n - y\|^2 : y \in C \right\} \\ z_n = \alpha_n \rho_n + (1 - \alpha_n) T t_n, \\ C_n = \{v \in C : \|w_n - v\| \leq \|\rho_n - v\|\} \\ Q_n = \{v \in C : \langle \pi_0 - \rho_n, v - \rho_n \rangle \leq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} \pi_0. \end{array} \right. \quad (3)$$

In order to establish the strong convergence of the sequences $\{\rho_n\}$ generated by (3) to $x^* = P_{EP(g) \cap F(T)} \pi_0$, the author required that the positive sequences $\{\sigma_n\}$ of the stepsize satisfies a Lipschitz-like condition, i.e.



$$0 < \sigma_n \leq \min \left\{ \frac{1}{2c_1}, \frac{1}{2c_2} \right\}, \quad (4)$$

where c_1 and c_2 are the Lipschitz-like constants of g .

In 2016, Hieu [7] following a similar trend, introduced the following Parallel Modified Extragradient Method (PMEM), for solving the finite family of equilibrium problems and the common fixed point of nonexpansive mappings in the framework of real Hilbert spaces:

$$\begin{cases} y_n^i = \operatorname{argmin} \left\{ \rho g_i(\rho_n, y) + \frac{1}{2} \|\rho_n - y\|^2 : y \in C \right\} & i = 1, \dots, N, \\ z_n^i = \operatorname{argmin} \left\{ \rho g_i(y_n^i, y) + \|\rho_n - y\|^2 : y \in C \right\} & i = 1, \dots, N, \\ i_n \in \operatorname{Argmax} \left\{ \|z_n^i - \rho_n\| : i = 1, 2, \dots, N \right\}, & \bar{z}_n := z_n^{i_n}, \\ j_n \in \operatorname{Argmax} \left\{ \|u_n^j - \rho_n\| : j = 1, 2, \dots, M \right\} \\ C_n = \{v \in C : \|\bar{u}_n - v\| \leq \|\rho_n - v\|\}, \\ Q_n = \{v \in C : \langle \pi_0 - \rho_n, v - \rho_n \rangle \leq 0\}, \\ \rho_{n+1} = P_{C_n \cap Q_n} \pi_0. \end{cases}$$



(5)

They also proved a strong convergence result for the sequence $\{\rho_n\}$ generated by (5) provided the stepsize ρ satisfies the following condition:

$$0 < \rho < \min \left\{ \frac{1}{2c_1}, \frac{1}{2c_2} \right\}, \quad (6)$$

where $c_1 = \max\{c_{1,i} : i = 1, 2, \dots, N\}$ and $c_2 = \max\{c_{2,i} : i = 1, 2, \dots, N\}$, $c_{1,i}$ and $c_{2,i}$ are the Lipschitz-like constants for g_i for $i = 1, 2, \dots, N$.

In 2014, Shahzad and Zegeye [3], in the framework of real reflexive Banach spaces and for multi-valued Bregman relatively nonexpansive mappings, presented the following iterative scheme for the common fixed points:

$$\begin{cases} u, \pi_0 \in X, \beta_n = \text{Proj}_C^f \nabla f(u) + (1 - \alpha_n) \nabla f(\rho_n), \\ \rho_{n+1} = \nabla f * (w_n) + \sum_{i=1}^N \beta_i \nabla f(u_{i,n}), \end{cases}$$

where $u_{i,n} \in T_i w_n$, C is a nonempty closed convex subset of $\text{int dom } f$. Under some mild conditions on the parameters, the authors proved the strong convergence of the sequence $\{\rho_n\}$ to



$Proj_F^f u$, where $F = \bigcap_{i=1}^N F(T_i)$.

Very recently, Eskandani [5], using the Hybrid Parallel extragradient method (HPA), introduced a Bregman–Lipschitz-type condition for a pseudomonotone bifunction. For estimating this common point X^* for a finite family of multi-valued Bregman relatively nonexpansive mappings in reflexive Banach spaces, the following algorithm, called HPA, was presented:

$$\begin{cases} \beta_n^i = \operatorname{argmax} \{ \sigma_n g_i(\rho_n, \beta) + D_f(\beta, \rho_n) : w \in C \} & i = 1, \dots, N, \\ z_n^i = \operatorname{argmin} \{ \sigma_n g_i(\beta_n^i, z) + D_f(z, \rho_n) : z \in C \} & i = 1, \dots, N, \\ j_n \in \operatorname{argmax} \{ D_f(z_n^i, \rho_n), \quad j = 1, 2, \dots, N \}, & \bar{z}_n := z_n^{j_n}, \\ y_n = \nabla f^*(\beta_{n,0} \nabla f(\bar{z}_n)) + \sum_{r=1}^M \beta_{n,r} \nabla f(z_{n,r}), & z_{n,r} \in T_r \bar{z}_n, \\ \rho_{n+1} = Proj_C^f(\nabla f(u_n) + (1 - \alpha_n) \nabla f(y_n)), \end{cases}$$



where D_f is the related Bregman distance of a function f . Under specific suppositions, they established that the sequence $\{\rho_n\}$ converges strongly to $Proj_{\Omega}^f u$, where

$\Omega = \bigcap_{r=1}^M F(T_r) \cap \bigcap_{i=1}^N EP(g_i) \neq \emptyset$ provided that the stepsize σ_n satisfies the condition:

$$\{\sigma_n\} \subset [a, b] \subset (0, q), \quad \text{where} \quad q = \min \left\{ \frac{1}{c_1}, \frac{1}{c_2} \right\}, \quad (7)$$

$c_1 = \max_{1 \leq i \leq N} \{c_{i,1}\}$, $c_2 = \max_{1 \leq i \leq N} \{c_{i,2}\}$ and $c_{i,1}, c_{i,2}$ are the Bregman-Lipschitz coefficients of bifunction g_i for $i = 1, 2, \dots, N$.



Motivation of Main Results

It is worthy to note that the results of Anh, Hieu, Eskandani, mentioned above, and other similar ones in the literature involves prior knowledge of the Lipschitz-like constants which have proven to be very strenuous to approximate practically. In fact, when it is possible, the estimates are often too small which slows down the rate of convergence of the algorithms. Thus, it becomes very important to find an algorithm which does not depend on the prior knowledge of the Lipschitz-like constants. Recently, many researchers introduced some modified extragradient algorithms for solving pseudomonotone EP (when $N = 1$) which does not involve prior estimate of the Lipschitz-like constants c_1 and c_2 .



Throughout this work, E and C be as defined in the beginning. Denote the dual space of E by E^* . Let $f : E \rightarrow (-\infty, \infty]$ be a proper convex and lower semicontinuous function. We denote the domain of f by $\text{dom } f$, which is the set $\{x \in E : f(x) < \infty\}$. Letting $x \in \text{int dom } f$, we define the subdifferential of the function f at x as the convex set such that

$$\partial f(x) = \{\varsigma \in E^* : f(x) + \langle y - x, \varsigma \rangle \leq f(y), \quad \forall y \in E\}. \quad (8)$$

We also define the Fenchel conjugate of f , as the function

$$f^* : E^* \rightarrow (-\infty, \infty], \quad \text{such that} \quad f^*(\varsigma) = \sup\{\langle \varsigma, x \rangle - f(x) : x \in E\}. \quad (9)$$



In 1967, Bregman [2] introduced Bregman distances, and in so doing he found an exquisite and efficacious tool for the utilization of the Bregman distance in the time spent designing and dissecting feasibility of optimization algorithms. Moving onwards, we presume $f : E \rightarrow (-\infty, \infty]$ is as also Legendre. The Bregman distance defined as the bifunction $D_f : \text{dom} f \times \text{int dom} f \rightarrow [0, +\infty)$, where

$$D_f(y, x) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle.$$

Looking at the Bregman distance through the lenses of a metric, it does not satisfy symmetry and the triangle inequality, hence it is not a metric. However, it generalizes the law of cosines, which we call the three point identity, it is as follows: for any $x \in \text{dom} f$ and $y, z \in \text{int dom} f$

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle. \quad (10)$$



Main results

Throughout this paper, we assume that the following assumptions hold on g :

- Ⓐ1 g is pseudomonotone, i.e., $g(x, y) \geq 0$ and $g(y, x) \leq 0$ for all $x, y \in C$,
- Ⓐ2 g is Bregman - Lipschitz - type condition, i.e, there exist two positive constants c_1, c_2 , such that

$$g(x, y) + g(y, z) \geq g(x, z) - c_1 D_f(y, x) - c_2 D_f(z, y), \quad \forall x, y, z \in C. \quad (11)$$

- Ⓐ3 $g(x, x) = 0$ for all $x \in C$,
- Ⓐ4 $g(\cdot, y)$ is continuous on C for every $y \in C$,
- Ⓐ5 $g(x, \cdot)$ is convex, lower semicontinuous, and subdifferentiable on C for every fixed $x \in C$.



$$\left\{ \begin{array}{l} \text{Pick } x_0 \in C, \mu \in (0, 1), \lambda_0 > 0, \text{ set } n = 0. \\ y_n^i = \operatorname{argmin} \left\{ g_i(x_n, y) + \frac{1}{\lambda_n} D_f(x_n, y) : y \in C \right\}, \\ z_n^i = \operatorname{argmin} \left\{ g_i(y_n^i, y) + \frac{1}{\lambda_n} D_f(x_n, y) : y \in T_n^i \right\}, \\ T_n^i = \{ z \in E : \langle \nabla f(x_n) - \lambda_n \omega_n^i - \nabla f(y_n^i), z - y_n^i \rangle \leq 0 \} \\ w_n^i \in \partial g_i(x_n, y_n^i). \\ C_n = \{ z \in E : D_f(z, \bar{u}_n) \leq D_f(z, x_n) \} \\ Q_n = \{ z \in E : \langle \nabla f(z) - \nabla f(x_n), x_n - x_0 \rangle \geq 0 \}. \end{array} \right.$$

(12)



Before we start with the proof of Algorithm 10, we discuss some contributions of the algorithm compared with other methods in the literature.

- Firstly, Algorithm 1 solves two strongly convex optimization problems in parallel for $i = 1, 2, \dots, N$, with the second convex problem solving over the half-spaces T_n^i , which is simpler than the entire feasible set used in Eskandani et al. [19].
- Moreover, the stepsize in Eskandami et al. [19] required finding the prior estimates of the Lipschitz-like constants of the finite bifunctions, which is very cumbersome for computation. Meanwhile, in Algorithm 1, the stepsize is chosen self-adaptively and does not require the prior estimates of the Lipschitz-like constant of the finite bifunctions.
- Furthermore, when E is the real Hilbert space, our Algorithm 1 improves the algorithms of [18,47–50] in the setting of real Hilbert spaces.
- Furthermore, when E is a real Hilbert space and $N = 1, M = 1$, our Algorithm improves and compliments many other results in the literature



Main Theorem

Let E be a real reflexive Banach space, C be a nonempty, closed and convex subset of E and function $f : E \rightarrow \mathbb{R}$ be a uniformly Fréchet differentiable function, which is coercive, Legendre, totally convex and bounded on subsets of E such that $C \subset \text{int}(\text{dom}f)$. For $i = 1, 2, \dots, N$, let $g_i : E \times E \rightarrow \mathbb{R}$ be a finite family of bifunctions satisfying Assumptions (A1)–(A5). Furthermore, for $j = 1, 2, \dots, M$, let $T_j : E \rightarrow E$ be a finite family of Bregman relatively nonexpansive mappings. Suppose that $Sol = (\cap_{i=1}^N EP(g_i)) \cap (\cap_{j=1}^M F(T_j)) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $0 < \alpha_n \leq \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n \leq b < 1$. Then, the sequences $\{x_n\}$, $\{y_n^i\}$, $\{z_n^i\}$ generated by Algorithm 10 converge strongly to a solution x^* , where $x^* = Proj_{Sol}^f x_0$



Numerical Results

PHBSEM with f^1	Iter.	15	15	15	16
	Time (s)	1.0343	1.7926	4.7522	2.6533
PHBSEM with f^2	Iter.	9	9	9	9
	Time (s)	0.8500	1.1863	3.5812	2.4625
HPA with f^1	Iter.	21	32	77	37
	Time (s)	1.1719	6.5847	38.7433	11.6689
HPA with f^2	Iter.	19	17	18	30
	Time (s)	1.2669	3.7743	9.0533	9.3293
HPEM	Iter.	26	27	28	29
	Time (s)	1.4189	5.1728	12.5232	8.0773

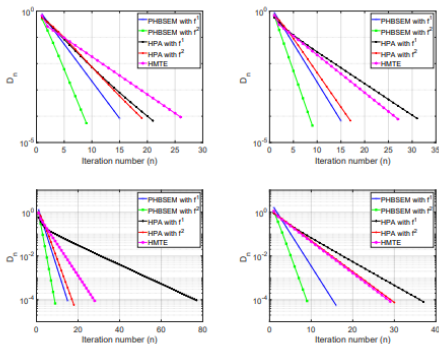


Figure 1. Example 1. Top left: I; top right: Case II; bottom left: Case III; bottom right: Case IV.



Concluding Remarks

In this paper, we present a new parallel hybrid Bregman subgradient extragradient method for finding a common solution of a finite family of the pseudomonotone equilibrium problem and common fixed point problem for Bregman relatively nonexpansive mappings in real Hilbert spaces. The algorithm is designed such that its convergence does not require prior estimates of the Lipschitz-like constant of the pseudomonotone bifunctions. Furthermore, a strong convergence result is proven under mild conditions. Some numerical examples are presented to show the efficiency and accuracy of the proposed method. This result improves and extends the results of and many other results in the literature.



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