NUMERICAL SOLUTION FOR A VIBRATING HIGH VOLTAGE OVERHEARD TRANSMISSION LINE

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INTRODUCTION

This study was inspired by the works of Prof AV Manzhirov and his team at the Russian Academy of Sciences, C Onunka and EE Ojo, MY Shatalov, and Sadiku & Obiozor.

A homogeneous rod of unit length with fixed ends is considered to represent a high voltage transmission line.

Hamilton’s variational principle is used to derive the equation of motion without damping terms.

Then Rayleigh’s dissipation energy function will be introduced to yield the equation of motion for high voltage transmission lines wind damping terms.

To obtain solutions to the resulting partial differential equations (PDE) in this study, a change of variables is introduced.

Thereafter, the Garlekin-Kantorovich method is used to find solutions.

Scaling will be introduced for both the equation of motions with and without damping, then their numerical solutions will be found.

Numerical method of lines (MOL) will be used to approximate the numerical solutions to the resulting PDEs.

This method converts a PDE into a system of coupled ordinary differential equations (ODE) of initial value type by using finite difference equations.

All results and graphs have been obtained through MATHEMATICA.

Derivation of the Equation of Motion for Vibrating High Voltage Transmission Lines

Let $u(t,x)$ be the displacement of particles within the rod in an axial direction between points along the rod, where the rod represents a power line.

The Lagrange equation is given by

$$L = T - P$$

…………………..(1)

where kinetic energy emitted or supplied due to the displacement of particles inside the rod is

$$T = \frac{1}{2}\int_0^l \rho A(x)\left(\frac{\partial u}{\partial t}\right)^2 dx, \qquad \text{....................(2)}$$

potential (strain) energy stored in the rod

$$P = \frac{1}{2}\int_0^l \left[EI\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + S\left(\frac{\partial u}{\partial x}\right)^2\right] dx, \qquad \text{..................(3)}$$

Then the Lagrangian is

$$L = L(\dot{u}, u'', u')$$

Then the Lagrangian density is

$$\Lambda = \Lambda\left(\dot{u}, u'', u'\right) \qquad \text{.....................(4)}$$

and the action integral is given by

$$A = \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2}\int_{0}^{l} \Lambda\left(\dot{u}, u'', u'\right) dx\, dt \qquad \text{..........(5)}$$

The variational principle for a monogenic system states that the action integral has a stationary point for the actual path motion

$$\delta A = \int_{t_1}^{t_2}\int_{0}^{l} \delta\Lambda\left(\dot{u}, u'', u'\right) dx\, dt = 0 \qquad \text{............(6)}$$

Then

$$\delta\Lambda(\dot{u},u'',u') = \frac{\partial\Lambda}{\partial\dot{u}}\delta\dot{u} + \frac{\partial\Lambda}{\partial u''}\delta u'' + \frac{\partial\Lambda}{\partial u'}\delta u' \quad \text{............(7)}$$

$$\Rightarrow \delta\Lambda(\dot{u},u'',u') = \frac{\partial}{\partial t}\left(\frac{\partial\Lambda}{\partial\dot{u}}\delta u\right) - \frac{\partial}{\partial t}\left(\frac{\partial\Lambda}{\partial\dot{u}}\right)\delta u + \frac{\partial}{\partial x}\left(\frac{\partial\Lambda}{\partial u''}\delta u'\right)$$

$$- \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial\Lambda}{\partial u''}\right)\delta u\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial\Lambda}{\partial u''}\right)\delta u$$

$$+ \frac{\partial}{\partial t}\left(\frac{\partial\Lambda}{\partial u'}\delta u\right) - \frac{\partial}{\partial t}\left(\frac{\partial\Lambda}{\partial u'}\right)\delta u$$

$$\text{.......(8)}$$

The variation of the action integral becomes:

$$
\begin{aligned}
\delta A =& \int_{t_1}^{t_2}\int_0^l \left[\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}} \delta u \right) - \frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}} \right) \delta u + \frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u''} \delta u' \right) \right] dx\, dt \\
&+ \int_{t_1}^{t_1}\int_0^l \left[- \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u''} \right) \delta u \right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \Lambda}{\partial u''} \right) \delta u \right] dx\, dt \\
&+ \int_{t_1}^{t_1}\int_0^l \left[\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u'} \delta u \right) - \frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u'} \right) \delta u \right] dx\, dt = 0
\end{aligned}
$$

....(9)

$$\delta A = \int_0^l \left(\frac{\partial \Lambda}{\partial \dot{u}} \delta u \right)_{t_1}^{t_2} dx - \int_{t_1}^{t_2}\int_0^l \frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}} \right) \delta u \, dx \, dt$$

$$+ \int_{t_1}^{t_2} \left(\frac{\partial \Lambda}{\partial u''} \delta u' \right)_0^l dt - \int_{t_1}^{t_2} \left(\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u''} \right) \delta u \right)_0^l dt$$

$$+ \int_{t_1}^{t_1}\int_0^l \frac{\partial^2}{\partial x^2}\left(\frac{\partial \Lambda}{\partial u''} \right) \delta u \, dx \, dt + \int_{t_1}^{t_2} \left(\frac{\partial \Lambda}{\partial u'} \delta u \right)_0^l dt$$

$$- \int_{t_1}^{t_1}\int_0^l \frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u'} \right) \delta u \, dx \, dt = 0 \qquad \text{......(10)}$$

Since the variation has stationary points at the actual path, then

$$\delta(x, t_1) = \delta(x, t_2) = 0 \quad \forall x \in [0, l]$$

Thus

$$\delta A = \int_{t_1}^{t_2}\int_{0}^{l}\left[-\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial \Lambda}{\partial u''}\right)-\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u'}\right)\right]\delta u\, dx\, dt$$

$$+\int_{t_1}^{t_2}\left(\frac{\partial \Lambda}{\partial u''}\delta u'\right)\Bigg|_{0}^{l} dt+\int_{t_1}^{t_2}\left[\frac{\partial \Lambda}{\partial u'}-\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u''}\right)\right]_{0}^{l}\delta u\, dt \quad \text{.....(11)}$$

$$=0$$

Yielding

$$-\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial \Lambda}{\partial u''}\right)-\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u'}\right)=0 \quad \text{.....(12)}$$

Subject to the boundary conditions

$$\left(\frac{\partial \Lambda}{\partial u''}\right)_0^l = 0 \quad \text{or} \quad \left(\frac{\partial u(x,t)}{\partial x}\right)_0^l = 0 \quad \text{and} \quad \left[\frac{\partial \Lambda}{\partial u'}\right]_0^l = 0$$

$$\text{or} \quad \left[u(x,t)\right]_0^l = 0$$

Making use of Equations (2) and (3),

$$\Lambda = \rho A \left(\frac{\partial u}{\partial t}\right)^2 - EI\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - S\left(\frac{\partial u}{\partial x}\right)^2$$

as well as (12), yields the boundary conditions:

$$\frac{\partial \Lambda}{\partial u''} = -EI\frac{\partial^2 u}{\partial x^2}; \quad \frac{\partial \Lambda}{\partial u'} = -S\frac{\partial u}{\partial x}; \text{ and } \frac{\partial \Lambda}{\partial \dot{u}} = \rho A\frac{\partial u}{\partial t}$$

$$\Rightarrow \quad \left(\frac{\partial \Lambda}{\partial u''}\right)_0^l = \left(-EI\frac{\partial^2 u(x,t)}{\partial x^2}\right)_0^l = 0 \text{ or } \left(\frac{\partial u(x,t)}{\partial x}\right)_0^l = 0$$

$$\text{and } \left[\frac{\partial \Lambda}{\partial u'}\right]_0^l = \left[-S\frac{\partial u(x,t)}{\partial x}\right]_0^l = 0 \text{ or } \left[u(x,t)\right]_0^l = 0$$

BCs: $\frac{\partial^2 u(0,t)}{\partial x^2} = 0$ and $u(0,t) = 0$ or $\frac{\partial^2 u(l,t)}{\partial x^2} = 0$ and $u(l,t) = 0$

ICs: $u(x,0) = u_0$ and $\frac{\partial u(x,0)}{\partial t} = \dot{u}_0$

$$\frac{\partial \Lambda}{\partial u''} = -EI\frac{\partial^2 u}{\partial x^2}; \quad \frac{\partial \Lambda}{\partial u'} = -S\frac{\partial u}{\partial x}; \quad \text{and} \quad \frac{\partial \Lambda}{\partial \dot{u}} = \rho A\frac{\partial u}{\partial t}$$

$$\Rightarrow \quad -\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}}\right) = -\frac{\partial}{\partial t}\left(\rho A\frac{\partial u}{\partial t}\right) = -\rho A\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial \Lambda}{\partial u''}\right) = \frac{\partial^2}{\partial x^2}\left(-EI\frac{\partial^2 u}{\partial x^2}\right) \quad \text{and}$$

$$-\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u'}\right) = -\frac{\partial}{\partial x}\left(-S\frac{\partial u}{\partial x}\right) = S\frac{\partial^2 u}{\partial x^2}$$

Thus, Equation (13) is the equation of motion for vibrating transmission lines without damping.

$$\rho A\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right) - S\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{.......(13)}$$

The Lagrange – Rayleigh equation is given by

$$-\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial \Lambda}{\partial u''}\right)-\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u'}\right)=-\frac{\partial R}{\partial \dot{u}}-\frac{\partial^2}{\partial x^2}\left(\frac{\partial R}{\partial u''}\right) \quad \text{……(14)}$$

Since

$$R=\frac{1}{2}\left(C\left(\frac{\partial u}{\partial t}\right)^2+\beta I\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial t\,\partial x^2}\right)=R\left(\dot{u},u'',\dot{u}''\right)$$

Then

$$\frac{\partial R}{\partial \dot{u}}=C\frac{\partial u}{\partial t} \quad \text{and} \quad \frac{\partial^2}{\partial x^2}\left(\frac{\partial R}{\partial u''}\right)=\frac{\partial^2}{\partial x^2}\left(\beta I\frac{\partial^3 u}{\partial t\,\partial x^2}\right)$$

and the equation of motion with damping is

$$\rho A\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right)-S\frac{\partial^2 u}{\partial x^2}=-C\frac{\partial u}{\partial t}-\frac{\partial^2}{\partial x^2}\left(\beta I\frac{\partial^3 u}{\partial t\,\partial x^2}\right) \quad \text{……(15)}$$

NUMERICAL SOLUTION OF THE EQUATION OF MOTION

A numerical solution of the PDE in Eq (13)

$$\rho A \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 u(x,t)}{\partial x^2} \right) - S \frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

$$\frac{\partial^2 u(0,t)}{\partial x^2} = 0 = u(0,t), \quad \frac{\partial^2 u(l,t)}{\partial x^2} = 0 = u(l,t)$$

$$u(x,0) = u_0 \quad \text{and} \quad \frac{\partial u(x,0)}{\partial t} = \dot{u}_0$$

is sought where $u = u(x,t)$ is the displacement, ρ is mass density, A is a cylindrical cross section of the rod, EI is flexural rigidity of the power line, S is tensile force.

Assume the solution of Eq (13) to be an infinite sine series which satisfies the boundary conditions

$$u(x,t) = \sum_{m=1}^{\infty} v_m(t) \sin\left(\frac{\pi m x}{l}\right) \qquad \text{.......(16)}$$

Then a system of ODEs resulting from substituting Eq (16) into Eq (13) is:

$$\left(\frac{EI}{\rho A}\left(\frac{m\pi}{l}\right)^4 + \frac{S}{\rho A}\left(\frac{m\pi}{l}\right)^2\right) v_m(t) + v_m''(t) = 0 \qquad \text{......(17)}$$

Solving Eq (17) yields a harmonic solution

$$v_m(t) = C_1 \sin(\omega_m t) + C_2 \cos(\omega_m t) \qquad \text{.......(18)}$$

where eigenvalues are given by

$$\omega_m^2 = \frac{EI}{\rho A}\left(\frac{m\pi}{l}\right)^4 + \frac{S}{\rho A}\left(\frac{m\pi}{l}\right)^2 \qquad \text{......}$$

(19)

The parameters used to calculate the eigenvalues are that of a homogeneous power line with viscous and structural damping.

$$EI = 615.8 \text{ Nm}^2; \quad \rho A = 1.3027 \text{ kg/m}; \quad S = 27468.0 \text{ N}$$

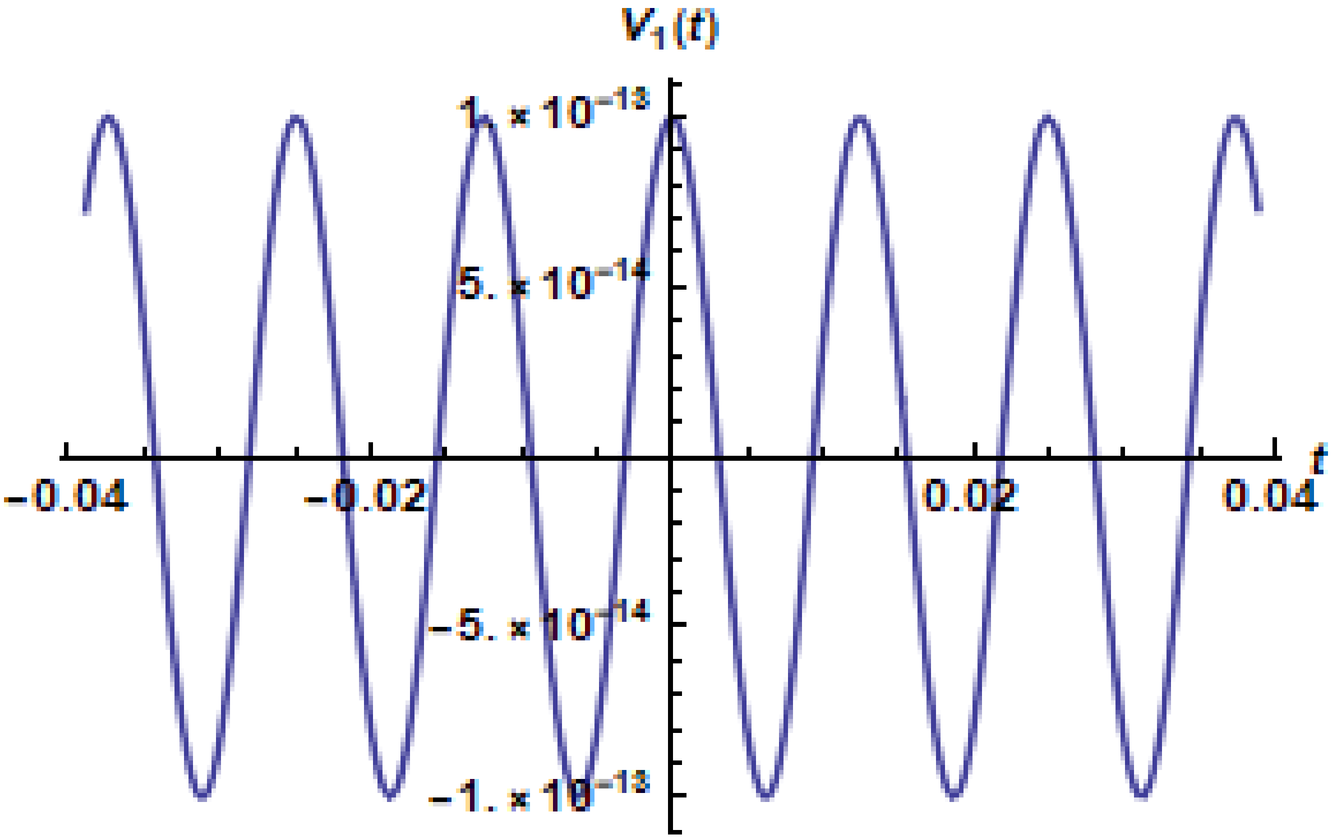
The initial conditions for this ODE system are:

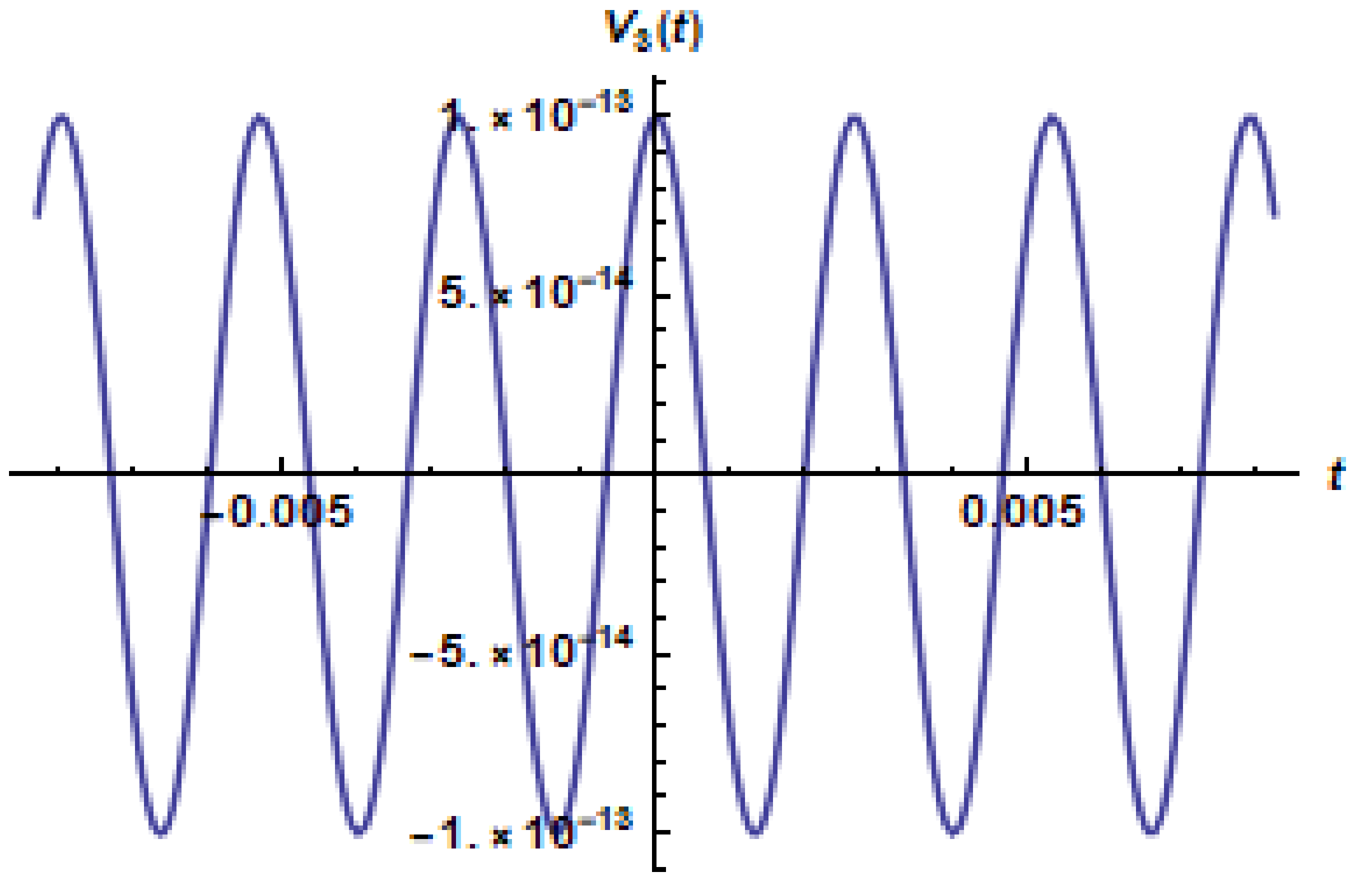
$$v_m(0) = 10^{-13} = \frac{dv_m(0)}{dt}$$

Yielding a particular solution:

$$v_m(t) = \frac{\sin(\omega_m t) + \omega_m \cos(\omega_m t)}{10^{13} \omega_m}$$

Graphs of which are shown below, for associated with the first and third eigenvalues:





V3(t)
1. × 10−13
5. × 10−14
−5. × 10−14
−1. × 10−13
−0.005
0.005
t

Garlekin-Kantorovich Method

A change of variables $t = \tau, \quad y = \dfrac{x}{1+\varepsilon t}$ is introduced on Eq (13) yielding

$$
-\left(\frac{\rho A y^2 \varepsilon^2}{(1+\varepsilon\tau)^2} + \frac{S}{(1+\varepsilon\tau)^2} \right) \frac{\partial^2 u(y,\tau)}{\partial y^2} + \rho A \frac{\partial^2 u(y,\tau)}{\partial \tau^2}
$$

$$
+ \frac{2\rho A y \varepsilon^2}{(1+\varepsilon\tau)^2} \frac{\partial u(y,\tau)}{\partial y} - \frac{2\rho A y \varepsilon}{(1+\varepsilon\tau)^2} \frac{\partial^2 u(y,\tau)}{\partial y \partial \tau}
$$

$$
+ \frac{EI}{(1+\varepsilon\tau)^4} \frac{\partial^4 u(y,\tau)}{\partial y^4} = 0
$$

Ignoring terms that have ε^2 since $0 < \varepsilon << 1$ then the equation to be solved is

$$-\frac{S}{(1+\varepsilon\tau)^2}\frac{\partial^2 u(y,\tau)}{\partial y^2}+\rho A\frac{\partial^2 u(y,\tau)}{\partial \tau^2}-\frac{2\rho A y\varepsilon}{(1+\varepsilon\tau)^2}\frac{\partial^2 u(y,\tau)}{\partial y\partial \tau}+\frac{EI}{(1+\varepsilon\tau)^4}\frac{\partial^4 u(y,\tau)}{\partial y^4}=0 \quad ....(20)$$

Substitute Eq (16) into Eq (20), multiply the resulting system of ODEs by $\sin\left(\frac{n\pi y}{l}\right)$, and then integrate w.r.t. y between 0 and 1, yields

$$\text{GarlekinSystem1} = \Big\{\frac{EI\,\pi^4\,V_1[\tau]}{2\,(1+\epsilon\,\tau)^4} + \frac{\pi^2\,S\,V_1[\tau]}{2\,(1+\epsilon\,\tau)^2} + \frac{\epsilon\,\rho A\,V_1'[\tau]}{2+2\,\epsilon\,\tau} + \frac{1}{2}\,\rho A\,V_1''[\tau] == 0,$$

$$\frac{8\,EI\,\pi^4\,V_2[\tau]}{(1+\epsilon\,\tau)^4} + \frac{2\,\pi^2\,S\,V_2[\tau]}{(1+\epsilon\,\tau)^2} + \frac{\epsilon\,\rho A\,V_2'[\tau]}{2+2\,\epsilon\,\tau} + \frac{1}{2}\,\rho A\,V_2''[\tau] == 0,$$

$$\frac{81\,EI\,\pi^4\,V_3[\tau]}{2\,(1+\epsilon\,\tau)^4} + \frac{9\,\pi^2\,S\,V_3[\tau]}{2\,(1+\epsilon\,\tau)^2} + \frac{\epsilon\,\rho A\,V_3'[\tau]}{2+2\,\epsilon\,\tau} + \frac{1}{2}\,\rho A\,V_3''[\tau] == 0,$$

$$\frac{128\,EI\,\pi^4\,V_4[\tau]}{(1+\epsilon\,\tau)^4} + \frac{8\,\pi^2\,S\,V_4[\tau]}{(1+\epsilon\,\tau)^2} + \frac{\epsilon\,\rho A\,V_4'[\tau]}{2+2\,\epsilon\,\tau} + \frac{1}{2}\,\rho A\,V_4''[\tau] == 0,$$

$$\frac{625\,EI\,\pi^4\,V_5[\tau]}{2\,(1+\epsilon\,\tau)^4} + \frac{25\,\pi^2\,S\,V_5[\tau]}{2\,(1+\epsilon\,\tau)^2} + \frac{\epsilon\,\rho A\,V_5'[\tau]}{2+2\,\epsilon\,\tau} + \frac{1}{2}\,\rho A\,V_5''[\tau] == 0,$$

$$\frac{648\,EI\,\pi^4\,V_6[\tau]}{(1+\epsilon\,\tau)^4} + \frac{18\,\pi^2\,S\,V_6[\tau]}{(1+\epsilon\,\tau)^2} + \frac{\epsilon\,\rho A\,V_6'[\tau]}{2+2\,\epsilon\,\tau} + \frac{1}{2}\,\rho A\,V_6''[\tau] == 0,$$

$$\frac{2401\,EI\,\pi^4\,V_7[\tau]}{2\,(1+\epsilon\,\tau)^4} + \frac{49\,\pi^2\,S\,V_7[\tau]}{2\,(1+\epsilon\,\tau)^2} + \frac{\epsilon\,\rho A\,V_7'[\tau]}{2+2\,\epsilon\,\tau} + \frac{1}{2}\,\rho A\,V_7''[\tau] == 0,$$

$$\frac{2048\,EI\,\pi^4\,V_8[\tau]}{(1+\epsilon\,\tau)^4} + \frac{32\,\pi^2\,S\,V_8[\tau]}{(1+\epsilon\,\tau)^2} + \frac{\epsilon\,\rho A\,V_8'[\tau]}{2+2\,\epsilon\,\tau} + \frac{1}{2}\,\rho A\,V_8''[\tau] == 0,$$

$$\frac{6561\,EI\,\pi^4\,V_9[\tau]}{2\,(1+\epsilon\,\tau)^4} + \frac{81\,\pi^2\,S\,V_9[\tau]}{2\,(1+\epsilon\,\tau)^2} + \frac{\epsilon\,\rho A\,V_9'[\tau]}{2+2\,\epsilon\,\tau} + \frac{1}{2}\,\rho A\,V_9''[\tau] == 0,$$

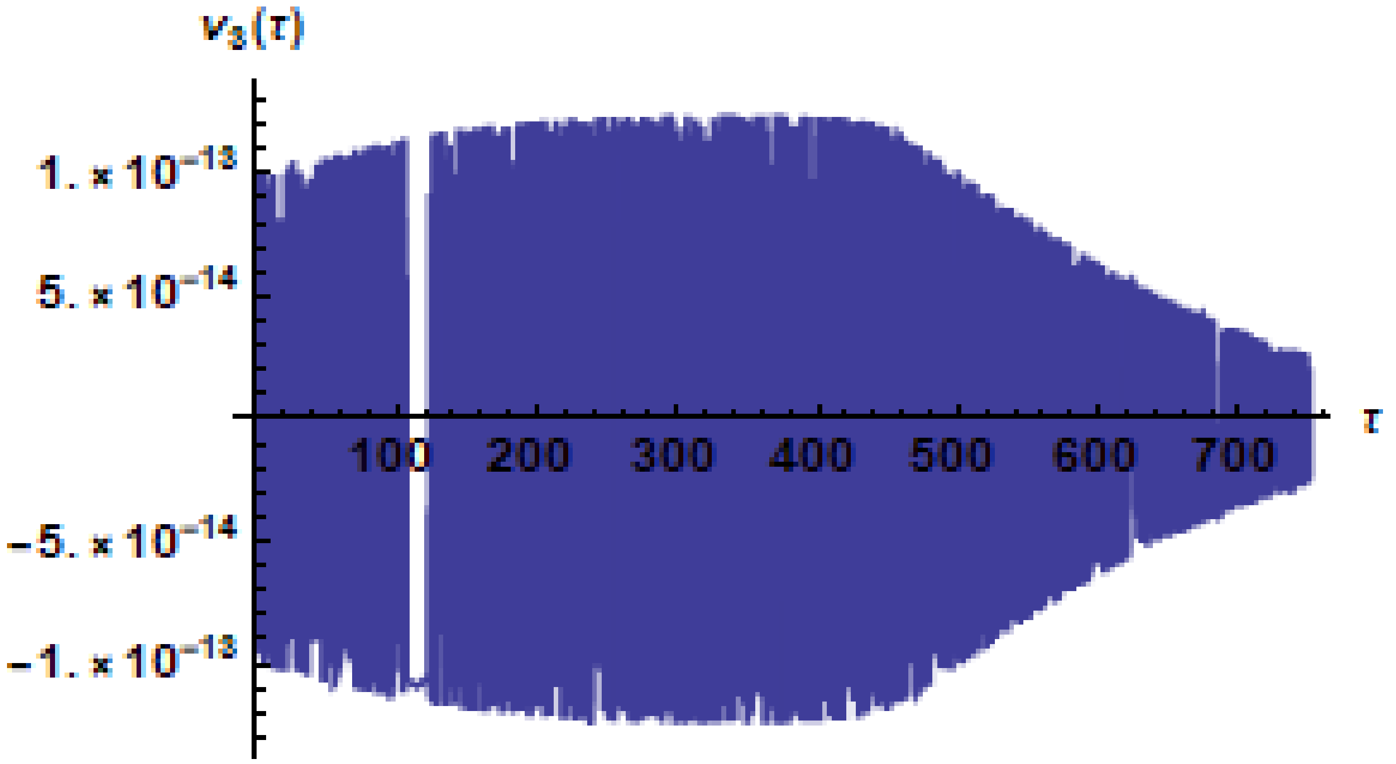
$$\frac{5000\,EI\,\pi^4\,V_{10}[\tau]}{(1+\epsilon\,\tau)^4} + \frac{50\,\pi^2\,S\,V_{10}[\tau]}{(1+\epsilon\,\tau)^2} + \frac{\epsilon\,\rho A\,V_{10}'[\tau]}{2+2\,\epsilon\,\tau} + \frac{1}{2}\,\rho A\,V_{10}''[\tau] == 0\Big\}$$

Making use of parameters and initial conditions

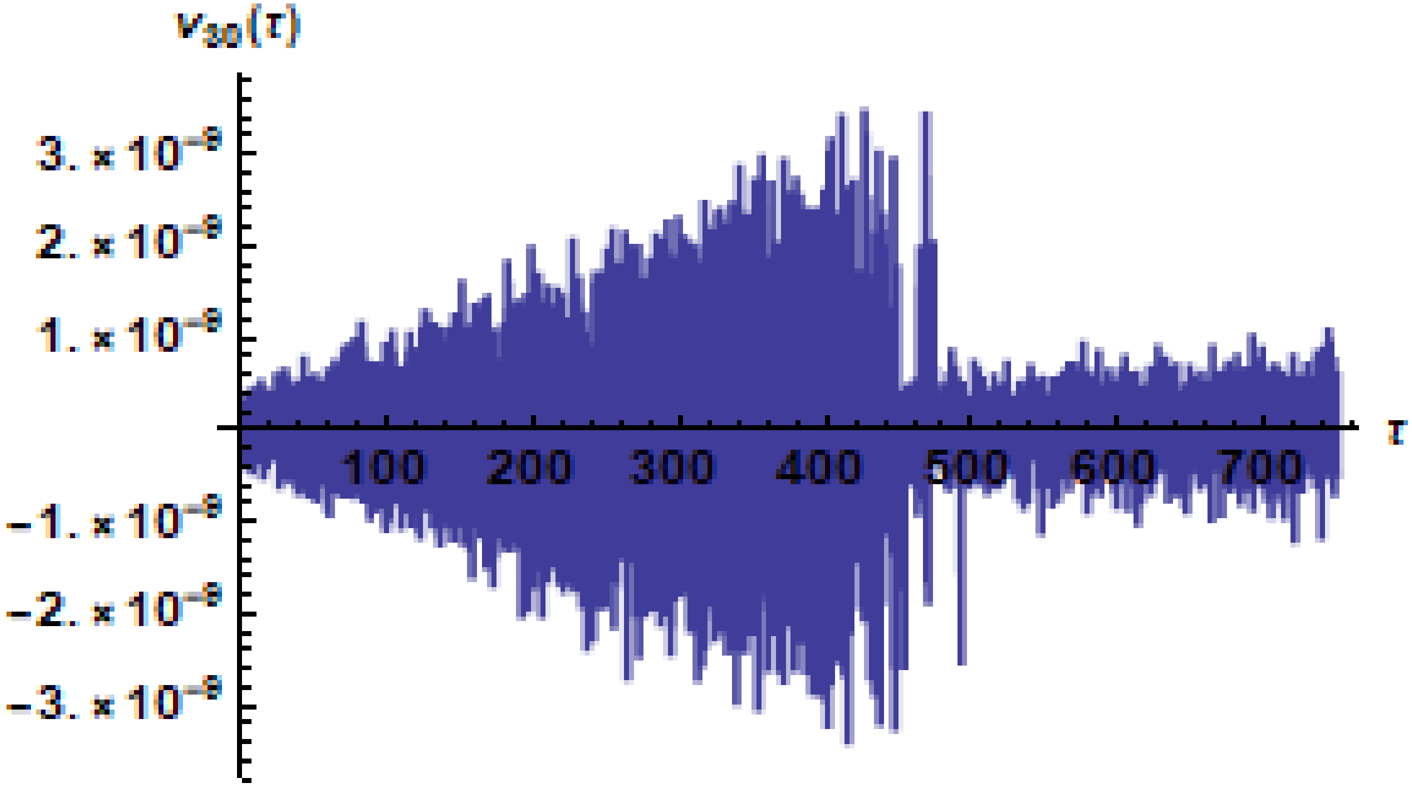
$$EI = 615.8 \text{ Nm}^2; \ \rho A = 1.3027 \text{ kg/m}; \ S = 27468.0 \text{ N}$$
$$\varepsilon = 0.005$$

to solve GarlekinSystem1 yields interpolating functions.

Plots of a few of these are as shown

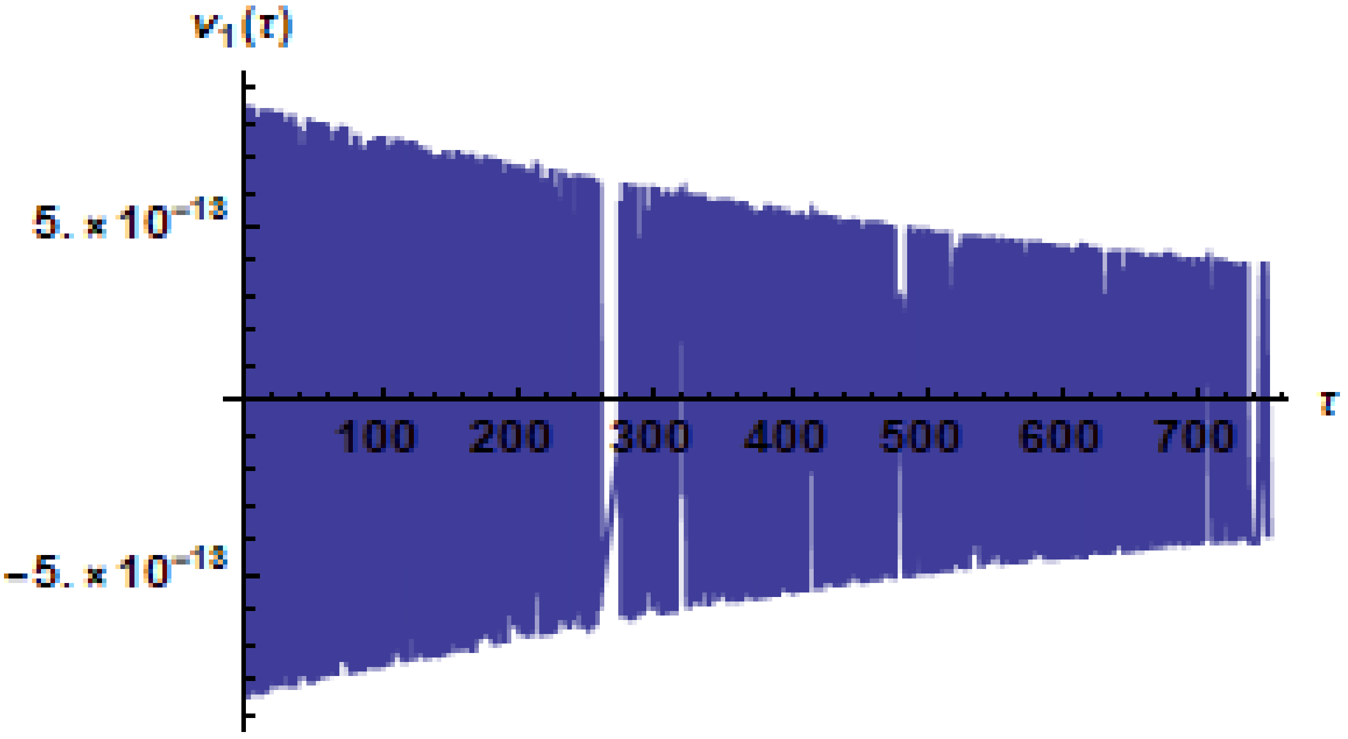


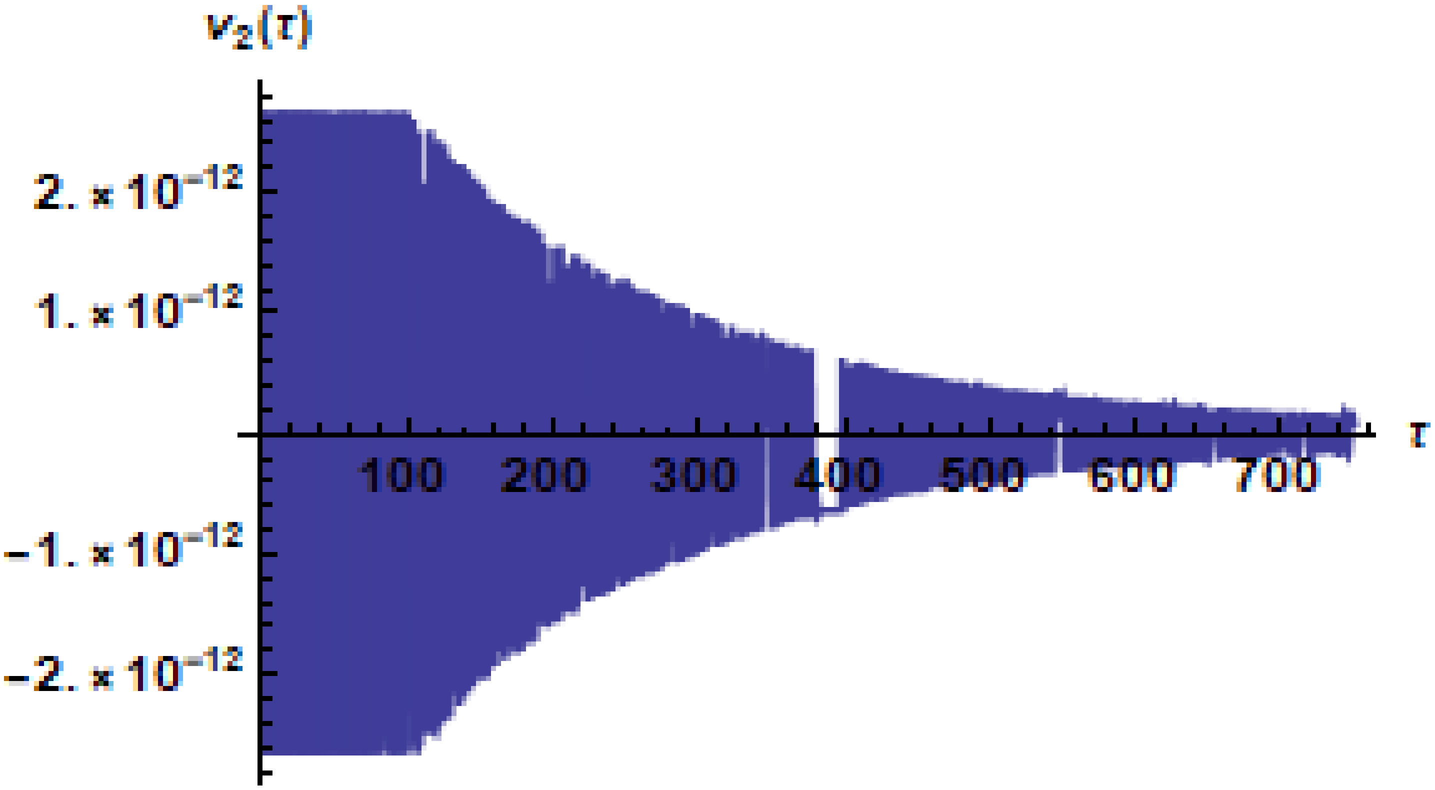
$v_3(\tau)$
$1. \times 10^{-13}$
$5. \times 10^{-14}$
$-5. \times 10^{-14}$
$-1. \times 10^{-13}$
100
200
300
400
500
600
700
τ



v30(τ)
3. × 10^-8
2. × 10^-8
1. × 10^-8
−1. × 10^-8
−2. × 10^-8
−3. × 10^-8
100
200
300
400
500
600
700
τ

With wind loading, results are





$v_2(\tau)$
$2. \times 10^{-12}$
$1. \times 10^{-12}$
$-1. \times 10^{-12}$
$-2. \times 10^{-12}$
100
200
300
400
500
600
700
τ

Change of variables on Lagrange-Rayleigh Eq (15) yields Eq (21):

$$\alpha\frac{\partial u(y,\tau)}{\partial\tau}+\left(\frac{2\rho A\varepsilon^{2}}{(1+\varepsilon\tau)^{2}}-\frac{y\,\alpha\,\varepsilon}{(1+\varepsilon\tau)}\right)\frac{\partial u(y,\tau)}{\partial y}$$

$$+\rho A\frac{\partial^{2}u(y,\tau)}{\partial\tau^{2}}+\left(\frac{\rho Ay^{2}\varepsilon^{2}}{(1+\varepsilon\tau)^{2}}-\frac{S}{(1+\varepsilon\tau)^{2}}\right)\frac{\partial^{2}u(y,\tau)}{\partial y^{2}}$$

$$-\frac{2\rho A\,y\,\varepsilon}{(1+\varepsilon\tau)}\frac{\partial^{2}u(y,\tau)}{\partial y\,\partial\tau}+\left(\frac{EI}{(1+\varepsilon\tau)^{4}}+\frac{4\beta I\varepsilon}{(1+\varepsilon\tau)^{5}}\right)\frac{\partial^{4}u(y,\tau)}{\partial y^{4}}$$

$$-\frac{\beta I}{(1+\varepsilon\tau)^{4}}\frac{\partial^{5}u(y,\tau)}{\partial y^{4}\partial\tau}=0 \qquad \text{......}$$

(21)

Substitute into Eq (21) the solution in Eq (16),

multiply by $\sin\left(\frac{n\pi y}{l}\right)$ **, then integrate w.r.t.** y **,**

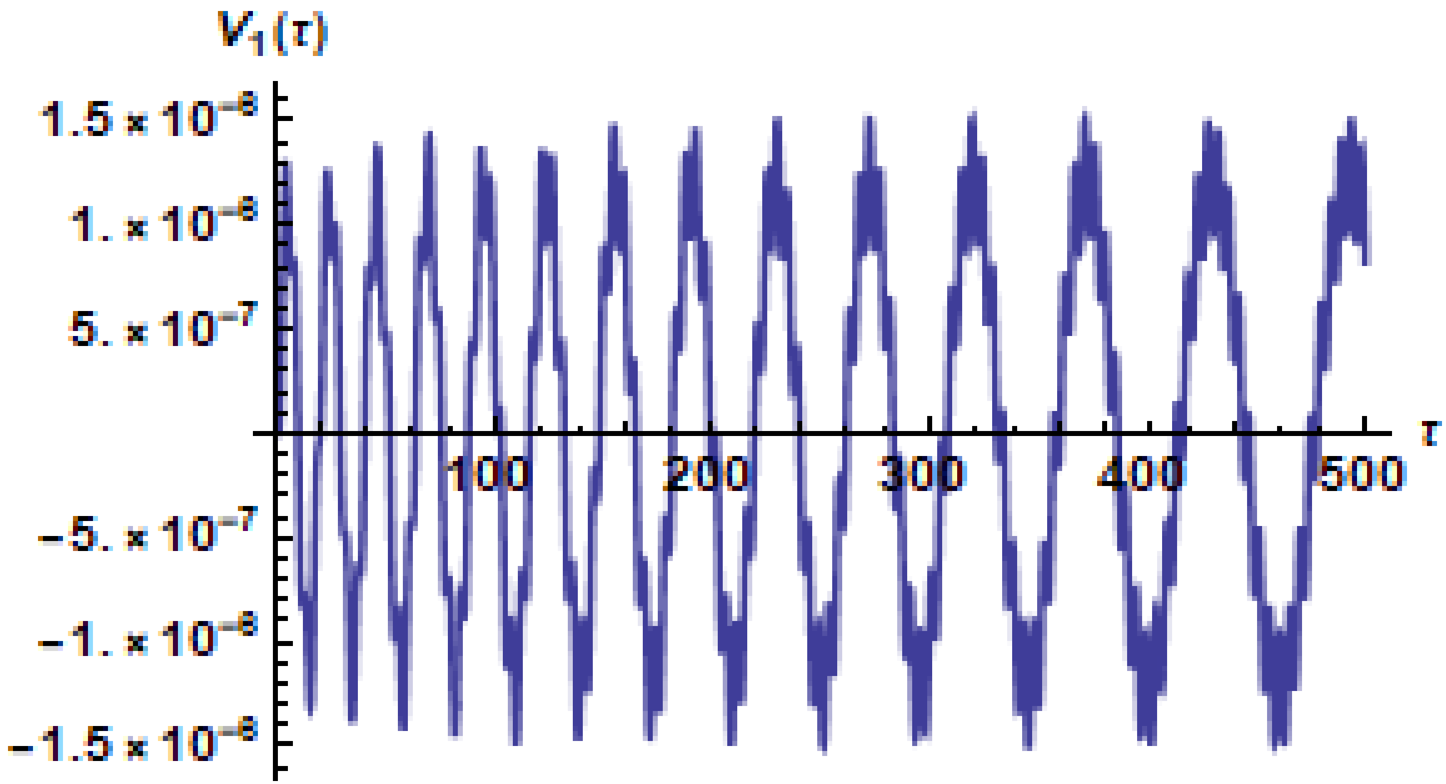
making use of parameters and initial conditions

$$EI = 615.8\ \text{Nm}^2;\ \rho A = 1.3027\ \text{kg/m};\ S = 27468.0\ \text{N};$$

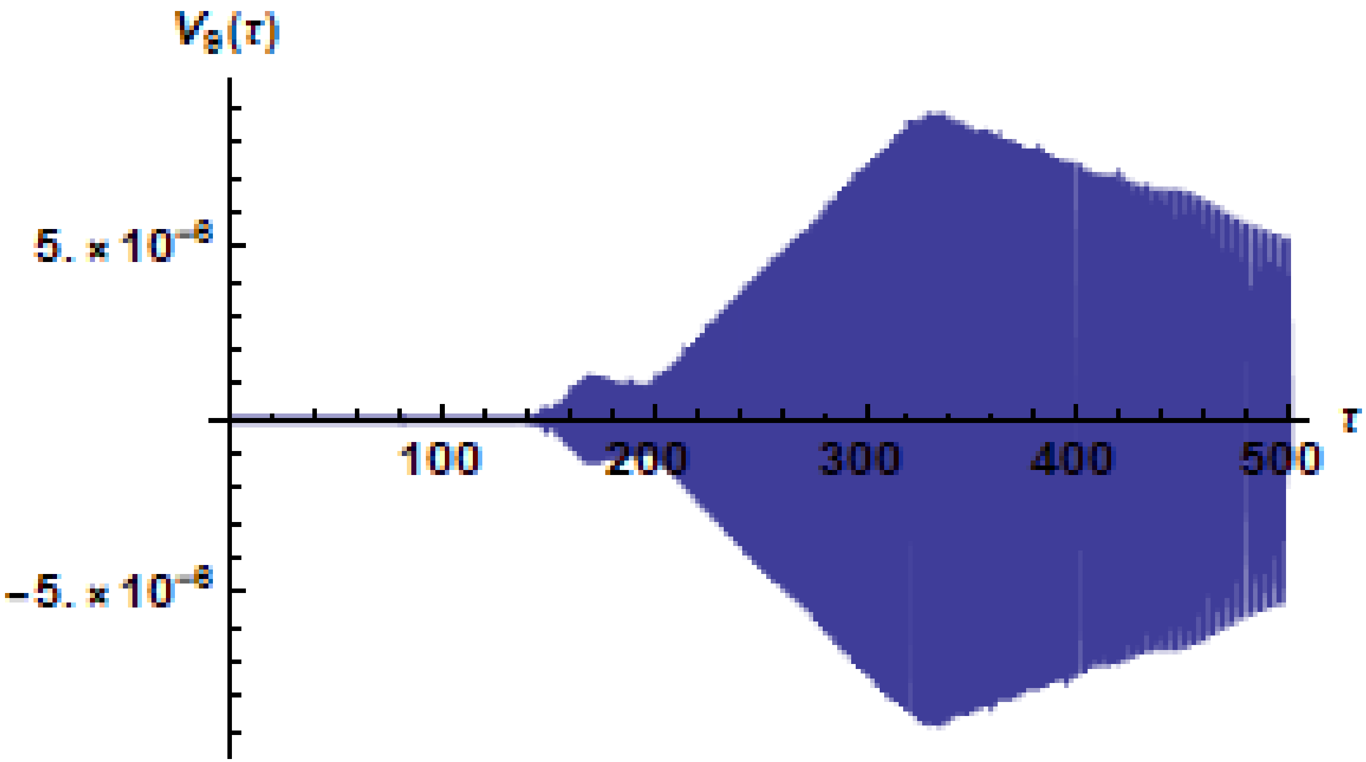
$$\varepsilon = 0.005;\ \alpha = 0.3674\ \text{Nm}^{-2}\text{s};\ \beta I = 0.028186\ \text{Nm}^2 s^{-1};$$

$$v_m(0) = 10^{-13} = \frac{dv_m(0)}{dt}$$

results in the plots:



$V_1(\tau)$
1.5 × 10^-8
1. × 10^-8
5. × 10^-7
−5. × 10^-7
−1. × 10^-8
−1.5 × 10^-8
100
200
300
400
500
τ



V8(τ)
5. × 10^-8
-5. × 10^-8
100
200
300
400
500
τ

Dimensionless scales are introduced to both Eq (13) and Eq (15):

$$t = T \cdot \tilde{t}$$

$$x = X \cdot \tilde{x}$$

$$l = X \cdot \tilde{l}$$

$$a = X \cdot \tilde{a}$$

$$u = U \cdot \tilde{u}$$

$$\tilde{u} = \tilde{u}\left(\tilde{t}, \tilde{x}\right) = U^{-1} \cdot u\left(t = T \cdot \tilde{t}, x = X \cdot \tilde{x}\right)$$

yielding

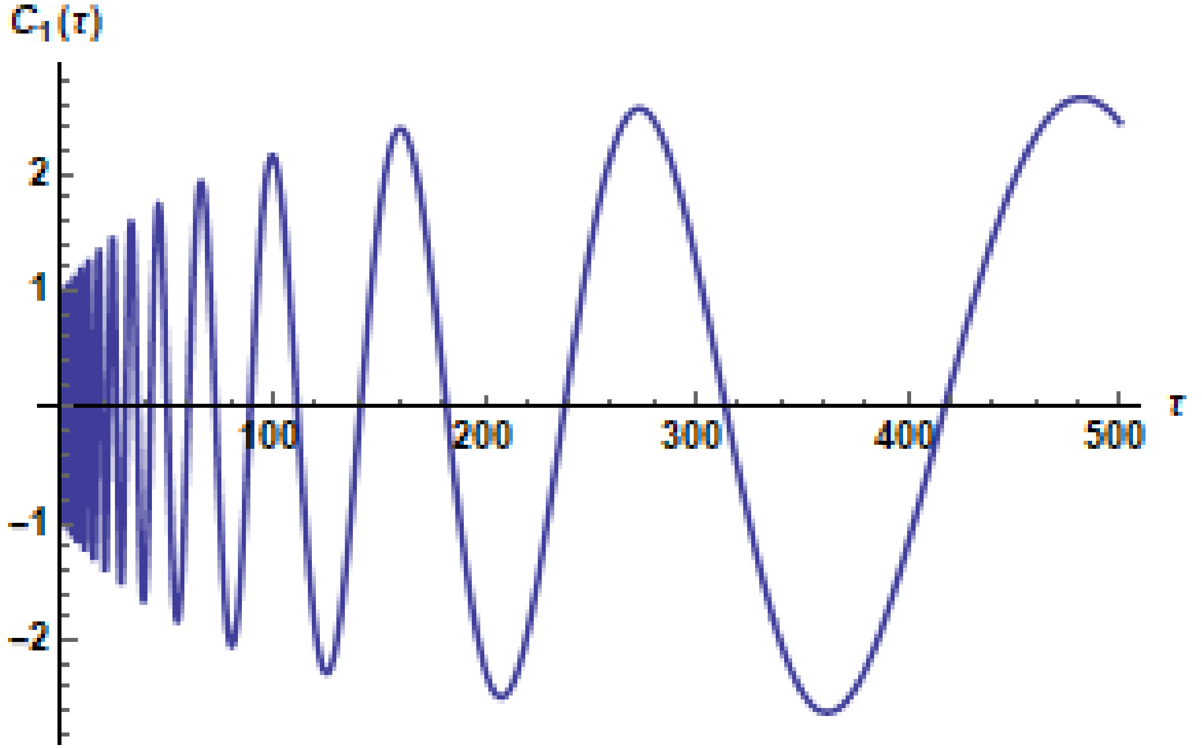
$$\frac{1}{2\pi}\frac{\partial^4 u(t,x)}{\partial t^4} - \xi\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{\partial^2 u(t,x)}{\partial t^2} = 0 \qquad \text{......(22)}$$

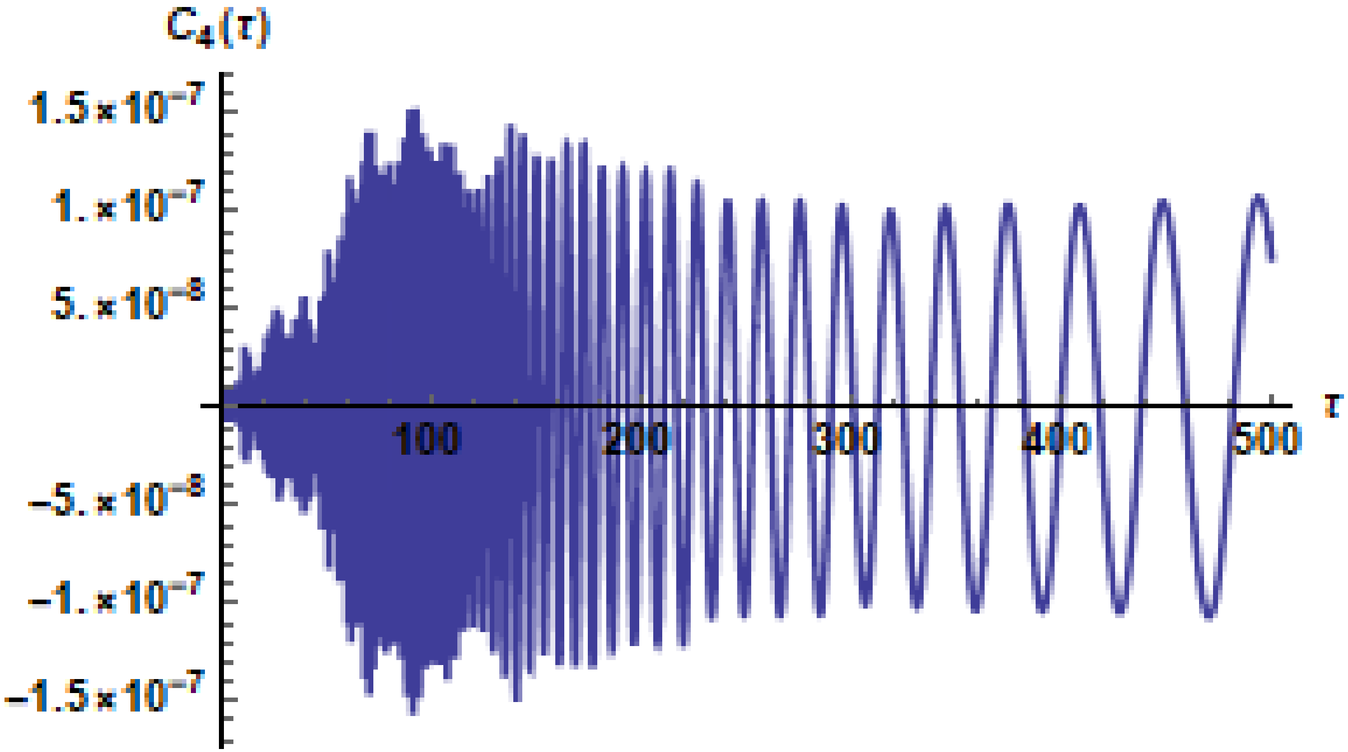
and

$$\frac{1}{2\pi}\frac{\partial^4 u(t,x)}{\partial t^4} - \xi\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{1}{2\pi}\frac{\partial^5 u(t,x)}{\partial x^4\partial t} + \frac{\partial u(t,x)}{\partial t} + \frac{\partial^2 u(t,x)}{\partial t^2} = 0 \qquad \text{.......(23)}$$

$$0 < \xi < 0.5222$$

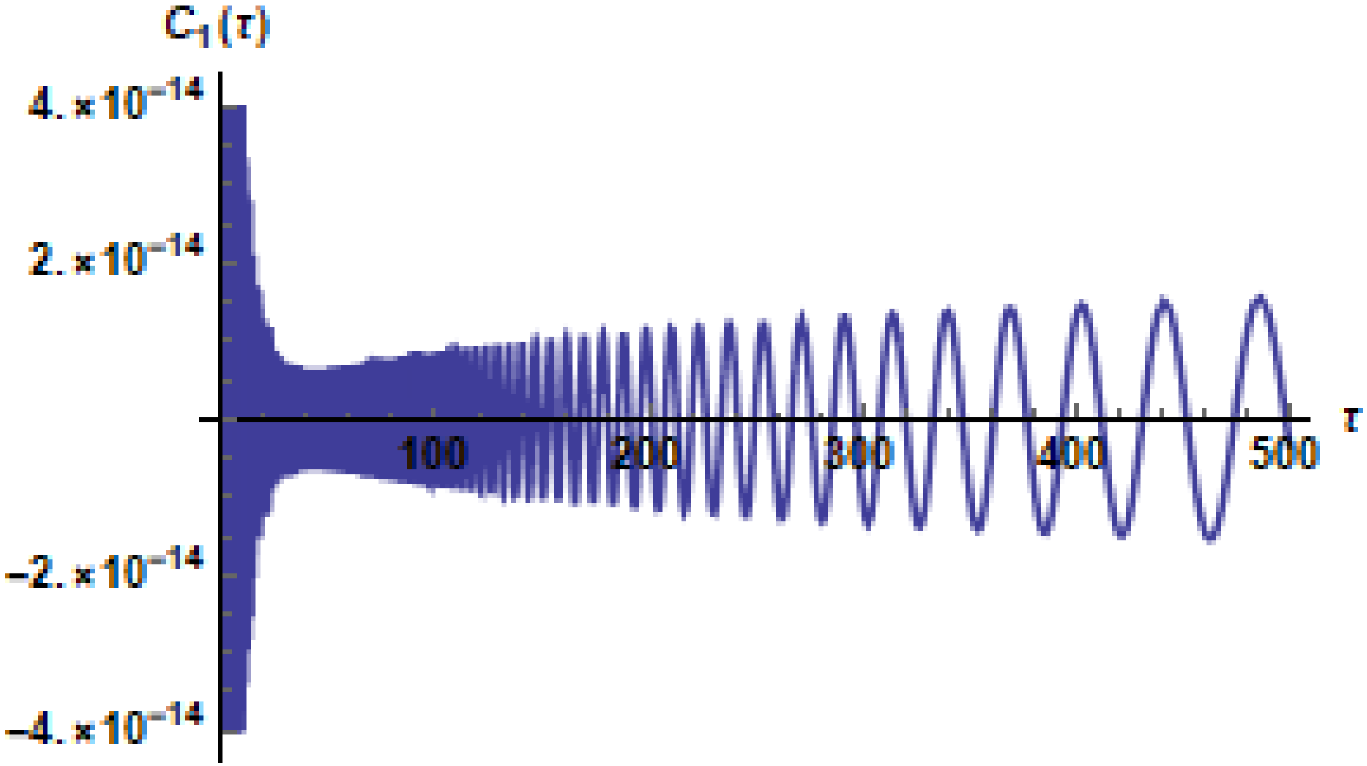
Plot of solution of the scaled equation of motion without damping:



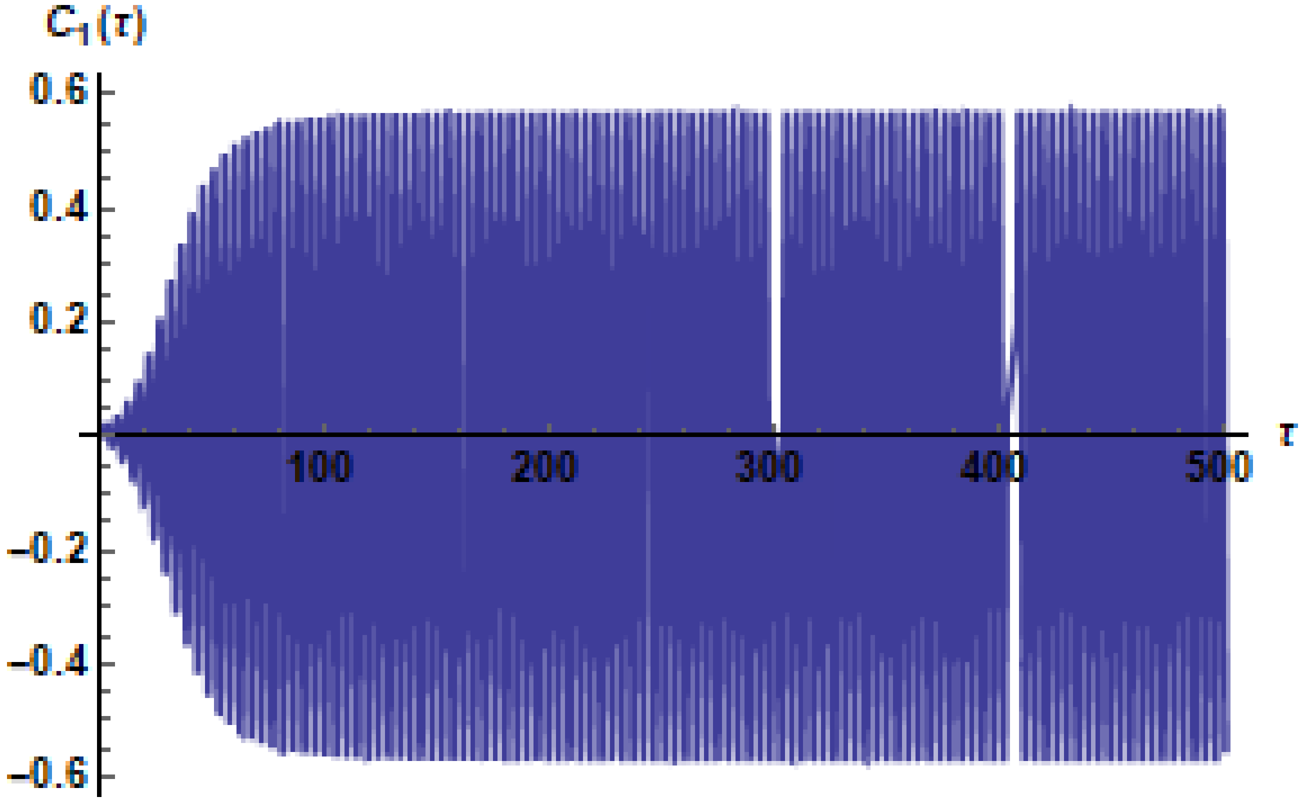


$C_4(\tau)$
1.5×10^{-7}
$1.\times10^{-7}$
$5.\times10^{-8}$
$-5.\times10^{-8}$
$-1.\times10^{-7}$
-1.5×10^{-7}
100
200
300
400
500
τ

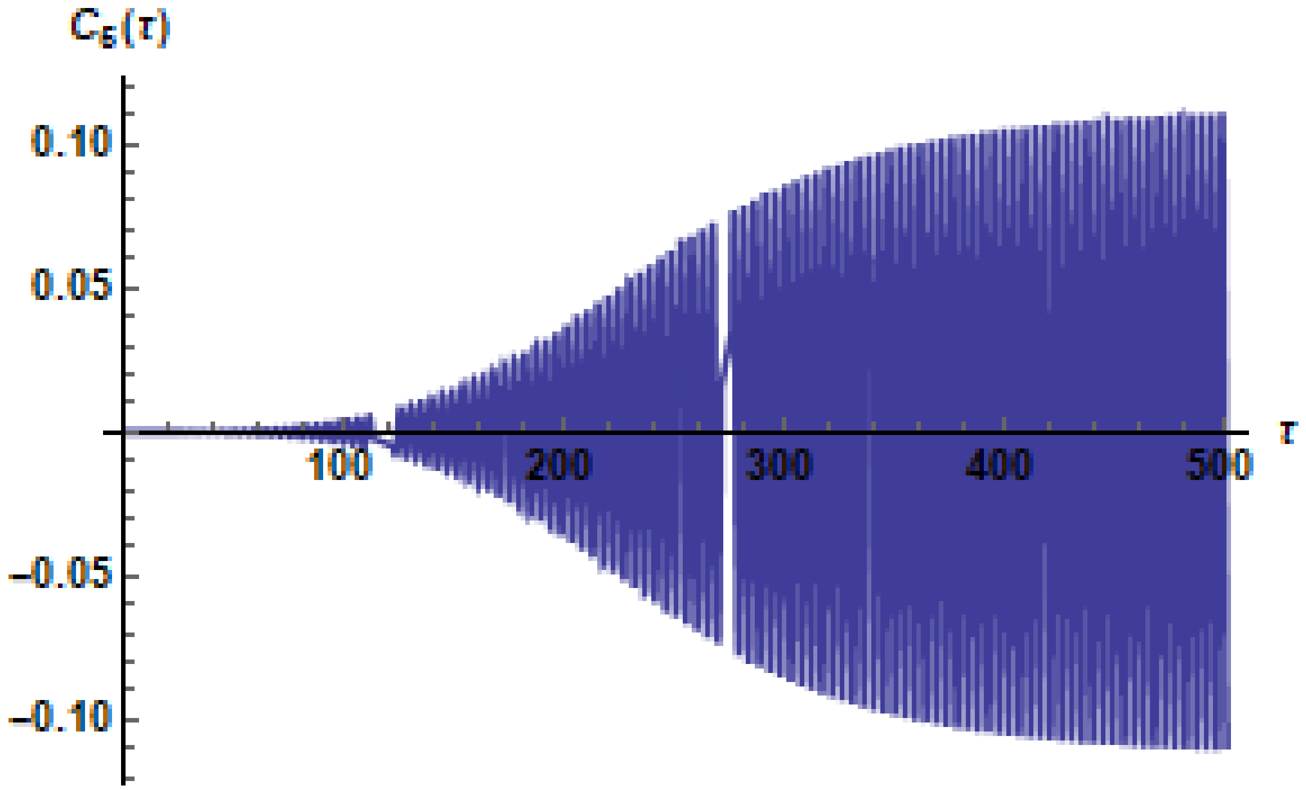
With wind load the result is



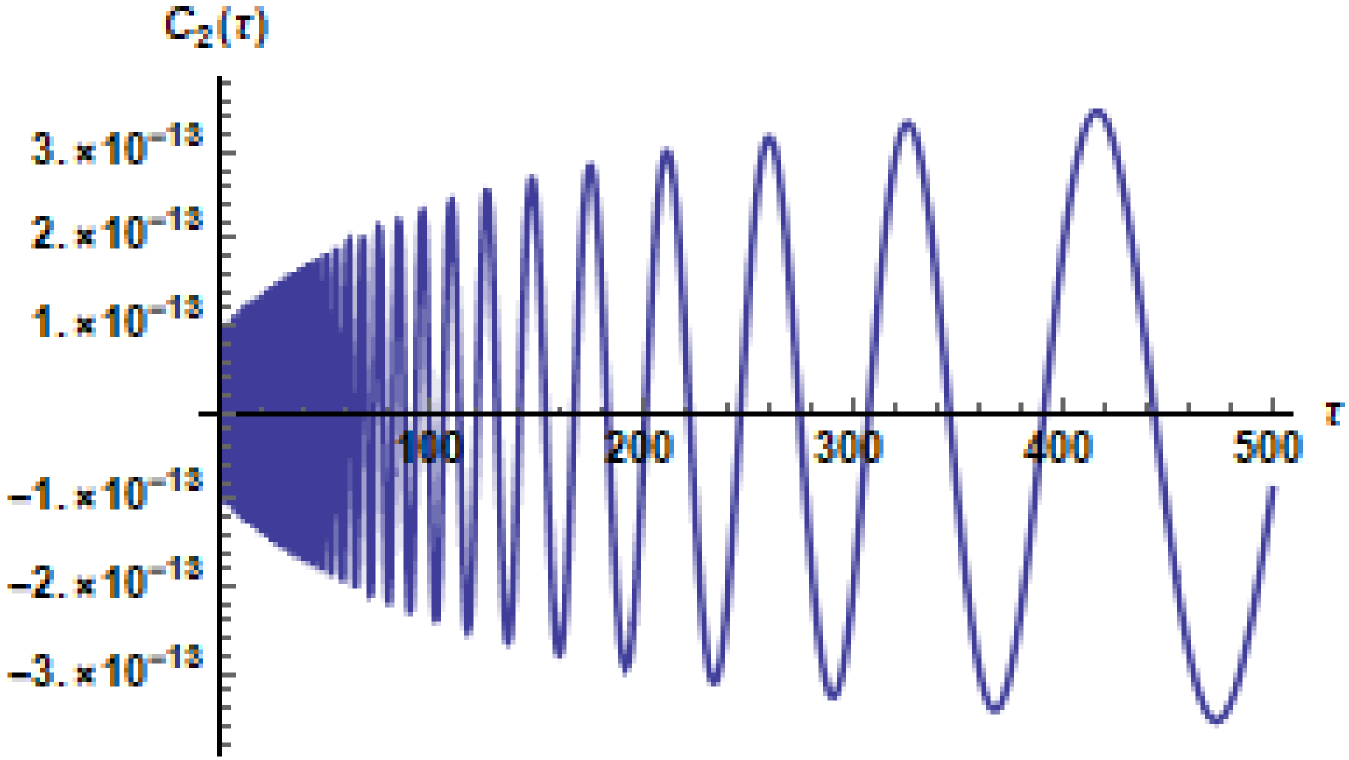
With Rayleigh's dissipation function:



With Rayleigh’s dissipation function:



With wind load



THE NUMERICAL METHOD OF LINES

When the finite difference equations are applied on a PDE, the interval is subdivided into a uniform grid.
For this study, the forward differences are used for the first grid, backward differences for the last grid, and central differences elsewhere after change of variables.

The resulting system of 11 coupled ODEs for $N = 10$ are shown:

MOLSystemNoDamp1 =

$$\left\{-\frac{\xi\,(2\,u_0[t]-5\,u_2[t]+4\,u_3[t]-u_4[t])}{h^2}+\frac{3\,u_0[t]-14\,u_1[t]+26\,u_2[t]-24\,u_3[t]+11\,u_4[t]-2\,u_5[t]}{2\,h^4\,\pi}+u_0''[t]==\mathrm{Sin}[t\,\omega]\,f_0,\right.$$

$$-\frac{\xi\,(u_0[t]-2\,u_1[t]+u_2[t])}{h^2}+\frac{u_{-1}[t]-4\,u_0[t]+6\,u_1[t]-4\,u_2[t]+u_3[t]}{2\,h^4\,\pi}+u_1''[t]==\mathrm{Sin}[t\,\omega]\,f_0,$$

$$-\frac{\xi\,(u_1[t]-2\,u_2[t]+u_3[t])}{h^2}+\frac{u_0[t]-4\,u_1[t]+6\,u_2[t]-4\,u_3[t]+u_4[t]}{2\,h^4\,\pi}+u_2''[t]==\mathrm{Sin}[t\,\omega]\,f_0,$$

$$-\frac{\xi\,(u_2[t]-2\,u_3[t]+u_4[t])}{h^2}+\frac{u_1[t]-4\,u_2[t]+6\,u_3[t]-4\,u_4[t]+u_5[t]}{2\,h^4\,\pi}+u_3''[t]==\mathrm{Sin}[t\,\omega]\,f_0,$$

$$-\frac{\xi\,(u_3[t]-2\,u_4[t]+u_5[t])}{h^2}+\frac{u_2[t]-4\,u_3[t]+6\,u_4[t]-4\,u_5[t]+u_6[t]}{2\,h^4\,\pi}+u_4''[t]==\mathrm{Sin}[t\,\omega]\,f_0,$$

$$-\frac{\xi\,(u_4[t]-2\,u_5[t]+u_6[t])}{h^2}+\frac{u_3[t]-4\,u_4[t]+6\,u_5[t]-4\,u_6[t]+u_7[t]}{2\,h^4\,\pi}+u_5''[t]==\mathrm{Sin}[t\,\omega]\,f_0,$$

$$-\frac{\xi\,(u_5[t]-2\,u_6[t]+u_7[t])}{h^2}+\frac{u_4[t]-4\,u_5[t]+6\,u_6[t]-4\,u_7[t]+u_8[t]}{2\,h^4\,\pi}+u_6''[t]==\mathrm{Sin}[t\,\omega]\,f_0,$$

$$-\frac{\xi\,(u_6[t]-2\,u_7[t]+u_8[t])}{h^2}+\frac{u_5[t]-4\,u_6[t]+6\,u_7[t]-4\,u_8[t]+u_9[t]}{2\,h^4\,\pi}+u_7''[t]==\mathrm{Sin}[t\,\omega]\,f_0,$$

$$-\frac{\xi\,(u_7[t]-2\,u_8[t]+u_9[t])}{h^2}+\frac{u_6[t]-4\,u_7[t]+6\,u_8[t]-4\,u_9[t]+u_{10}[t]}{2\,h^4\,\pi}+u_8''[t]==\mathrm{Sin}[t\,\omega]\,f_0,$$

$$-\frac{\xi\,(u_8[t]-2\,u_9[t]+u_{10}[t])}{h^2}+\frac{u_7[t]-4\,u_8[t]+6\,u_9[t]-4\,u_{10}[t]+u_{11}[t]}{2\,h^4\,\pi}+u_9''[t]==\mathrm{Sin}[t\,\omega]\,f_0,$$

$$\left.-\frac{\xi\,(-u_7[t]+4\,u_8[t]-5\,u_9[t]+2\,u_{10}[t])}{h^2}+\frac{-2\,u_5[t]+11\,u_6[t]-24\,u_7[t]+26\,u_8[t]-14\,u_9[t]+3\,u_{10}[t]}{2\,h^4\,\pi}+u_{10}''[t]==\mathrm{Sin}[t\,\omega]\,f_0\right\}$$

A trivial solution was found as expected since there was no wind load or input function.

$\{\{u_0[\tau] \to$ InterpolatingFunction[Domain: {{0., 500.}} Output: scalar] $[\tau]$,

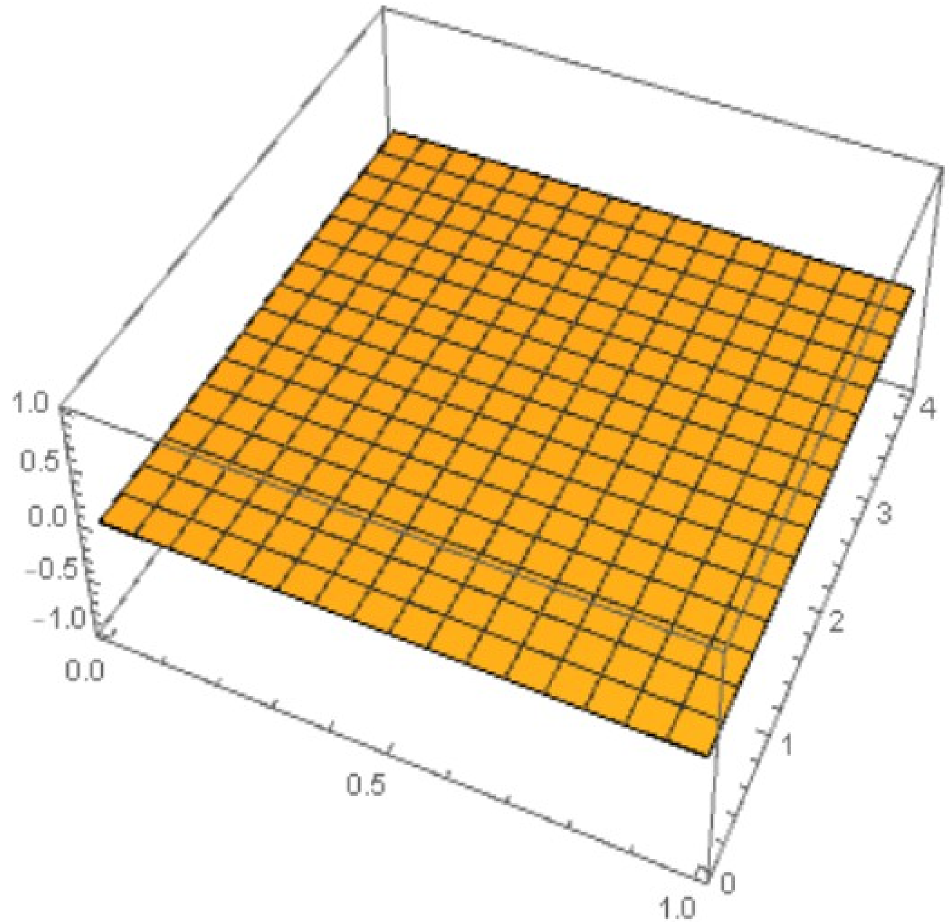
$u_1[\tau] \to$ InterpolatingFunction[Domain: {{0., 500.}} Output: scalar] $[\tau]$, $u_2[\tau] \to$ InterpolatingFunction[Domain: {{0., 500.}} Output: scalar] $[\tau]$,

$u_3[\tau] \to$ InterpolatingFunction[Domain: {{0., 500.}} Output: scalar] $[\tau]$, $u_4[\tau] \to$ InterpolatingFunction[Domain: {{0., 500.}} Output: scalar] $[\tau]$,

$u_5[\tau] \to$ InterpolatingFunction[Domain: {{0., 500.}} Output: scalar] $[\tau]$, $u_6[\tau] \to$ InterpolatingFunction[Domain: {{0., 500.}} Output: scalar] $[\tau]$,

$u_7[\tau] \to$ InterpolatingFunction[Domain: {{0., 500.}} Output: scalar] $[\tau]$, $u_8[\tau] \to$ InterpolatingFunction[Domain: {{0., 500.}} Output: scalar] $[\tau]$,

$u_9[\tau] \to$ InterpolatingFunction[Domain: {{0., 500.}} Output: scalar] $[\tau]$, $u_{10}[\tau] \to$ InterpolatingFunction[Domain: {{0., 500.}} Output: scalar] $[\tau]\}\}$


1.0
0.5
0.0
-0.5
-1.0
0.0
0.5
1.0
0
1
2
3
4

If we let $V_1(0) = 10^{-13}$ then MATHEMATICA output is shown below:

```
Out[ ]= {{V1[t] → InterpolatingFunction[ Domain: {{0., 500.}} Output: scalar ][t], V2[t] → InterpolatingFunction[ Domain: {{0., 500.}} Output: scalar ][t],
V3[t] → InterpolatingFunction[ Domain: {{0., 500.}} Output: scalar ][t], V4[t] → InterpolatingFunction[ Domain: {{0., 500.}} Output: scalar ][t],
V5[t] → InterpolatingFunction[ Domain: {{0., 500.}} Output: scalar ][t], V6[t] → InterpolatingFunction[ Domain: {{0., 500.}} Output: scalar ][t],
V7[t] → InterpolatingFunction[ Domain: {{0., 500.}} Output: scalar ][t], V8[t] → InterpolatingFunction[ Domain: {{0., 500.}} Output: scalar ][t],
V9[t] → InterpolatingFunction[ Domain: {{0., 500.}} Output: scalar ][t], V10[t] → InterpolatingFunction[ Domain: {{0., 500.}} Output: scalar ][t]}}
```

$$\{-100\,S\,(-5\,u_1[\tau]+4\,u_2[\tau]-u_3[\tau])+10000\,EI$$

$$\left(-14\,u_1[\tau]+26\,u_2[\tau]-24\,u_3[\tau]+\frac{3}{2}\,(5\,u_1[\tau]-4\,u_2[\tau]+u_3[\tau])+11\,u_4[\tau]-2\,u_5[\tau]\right)+$$

$$\rho A\,u_0''[\tau]==\text{Sin}\left[\frac{4\,\tau}{3}\right]\,f_0,\;-100\,S\,(-2\,u_1[\tau]+u_2[\tau])+$$

$$10000\,EI\,(7\,u_1[\tau]-4\,u_2[\tau]+u_3[\tau]-2\,(5\,u_1[\tau]-4\,u_2[\tau]+u_3[\tau]))+\rho A\,u_1''[\tau]==$$

$$\text{Sin}\left[\frac{4\,\tau}{3}\right]\,f_0,\;-100\,S\,(u_1[\tau]-2\,u_2[\tau]+u_3[\tau])+$$

$$10000\,EI\left(-4\,u_1[\tau]+6\,u_2[\tau]-4\,u_3[\tau]+\frac{1}{2}\,(5\,u_1[\tau]-4\,u_2[\tau]+u_3[\tau])+u_4[\tau]\right)+$$

$$\rho A\,u_2''[\tau]==\text{Sin}\left[\frac{4\,\tau}{3}\right]\,f_0,\;-100\,S\,(u_2[\tau]-2\,u_3[\tau]+u_4[\tau])+$$

$$10000\,EI\,(u_1[\tau]-4\,u_2[\tau]+6\,u_3[\tau]-4\,u_4[\tau]+u_5[\tau])+\rho A\,u_3''[\tau]==\text{Sin}\left[\frac{4\,\tau}{3}\right]\,f_0,$$

$$-100\,S\,(u_3[\tau]-2\,u_4[\tau]+u_5[\tau])+10000\,EI\,(u_2[\tau]-4\,u_3[\tau]+6\,u_4[\tau]-4\,u_5[\tau]+u_6[\tau])+$$

$$\rho A\,u_4''[\tau]==\text{Sin}\left[\frac{4\,\tau}{3}\right]\,f_0,\;-100\,S\,(u_4[\tau]-2\,u_5[\tau]+u_6[\tau])+$$

$$10000\,EI\,(u_3[\tau]-4\,u_4[\tau]+6\,u_5[\tau]-4\,u_6[\tau]+u_7[\tau])+\rho A\,u_5''[\tau]==\text{Sin}\left[\frac{4\,\tau}{3}\right]\,f_0,$$

$$-100\,S\,(u_5[\tau]-2\,u_6[\tau]+u_7[\tau])+10000\,EI\,(u_4[\tau]-4\,u_5[\tau]+6\,u_6[\tau]-4\,u_7[\tau]+u_8[\tau])+$$

$$\rho A\,u_6''[\tau]==\text{Sin}\left[\frac{4\,\tau}{3}\right]\,f_0,\;-100\,S\,(u_6[\tau]-2\,u_7[\tau]+u_8[\tau])+$$

$$10000\,EI\,(u_5[\tau]-4\,u_6[\tau]+6\,u_7[\tau]-4\,u_8[\tau]+u_9[\tau])+\rho A\,u_7''[\tau]==\text{Sin}\left[\frac{4\,\tau}{3}\right]\,f_0,$$

$$-100\,S\,(u_7[\tau]-2\,u_8[\tau]+u_9[\tau])+10000\,EI\,(u_6[\tau]-4\,u_7[\tau]+6\,u_8[\tau]-4\,u_9[\tau]+u_{10}[\tau])+$$

$$\rho A\,u_8''[\tau]==\text{Sin}\left[\frac{4\,\tau}{3}\right]\,f_0,\;10000\,EI\,(u_7[\tau]-4\,u_8[\tau]+7\,u_9[\tau]-4\,u_{10}[\tau])-$$

$$100\,S\,(u_8[\tau]-2\,u_9[\tau]+u_{10}[\tau])+\rho A\,u_9''[\tau]==\text{Sin}\left[\frac{4\,\tau}{3}\right]\,f_0,$$

$$-100\,S\,(-u_7[t]+4\,u_8[t]-5\,u_9[t]+2\,u_{10}[t])+$$

$$10000\,EI\,(-2\,u_5[\tau]+11\,u_6[\tau]-24\,u_7[\tau]+26\,u_8[\tau]-14\,u_9[\tau]+3\,u_{10}[\tau])+$$

$$\rho A\,u_{10}''[\tau]==\text{Sin}\left[\frac{4\,\tau}{3}\right]\,f_0\}$$

Challenges to Tackle

1. Continue working on applying MOL to PDEs of more than order 2.
2. Extend the methods exposed in this study to Rayleigh-Bishop and Mindlin-Hermann models.
3. An error analysis comparing the analytical solution to the MOL results to establish an error estimate.

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