# Coherent states in complex geometry with application to modular forms and representation theory


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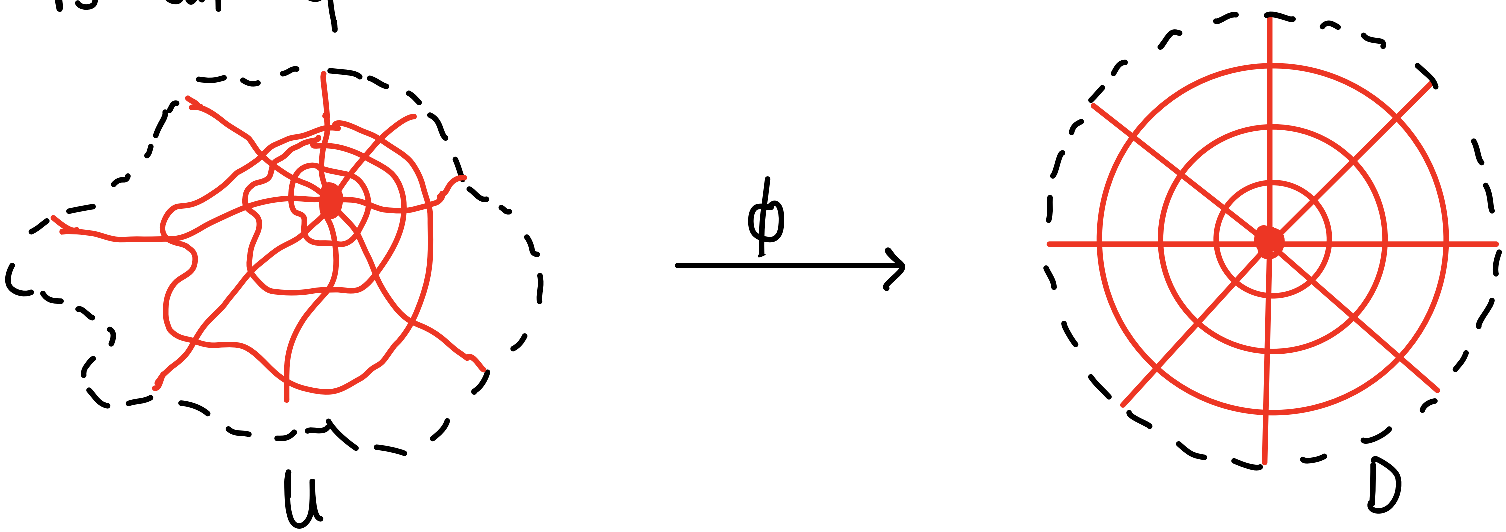
# 1. Introduction Recall:

<u>Riemann Mapping Theorem (1851)</u> Let $U \subset \mathbb{C}$ be a simply-connected open set. Then there exists a holomorphic bijection

$$\phi : U \xrightarrow{\cong} D$$

where $D$ is an open disk.



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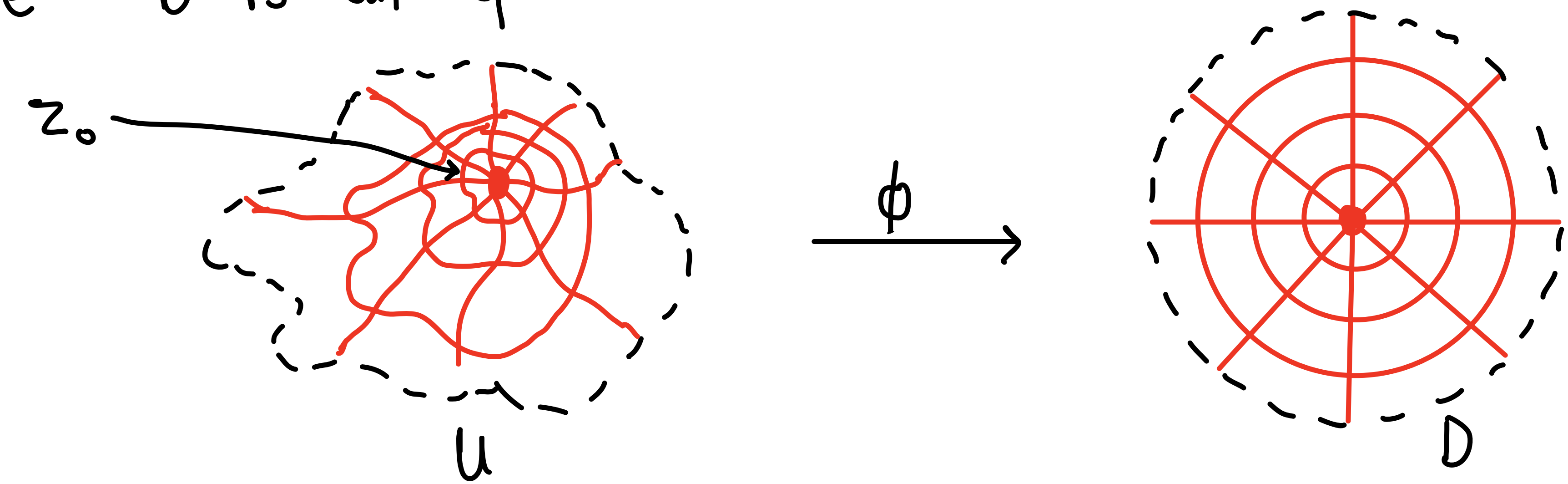


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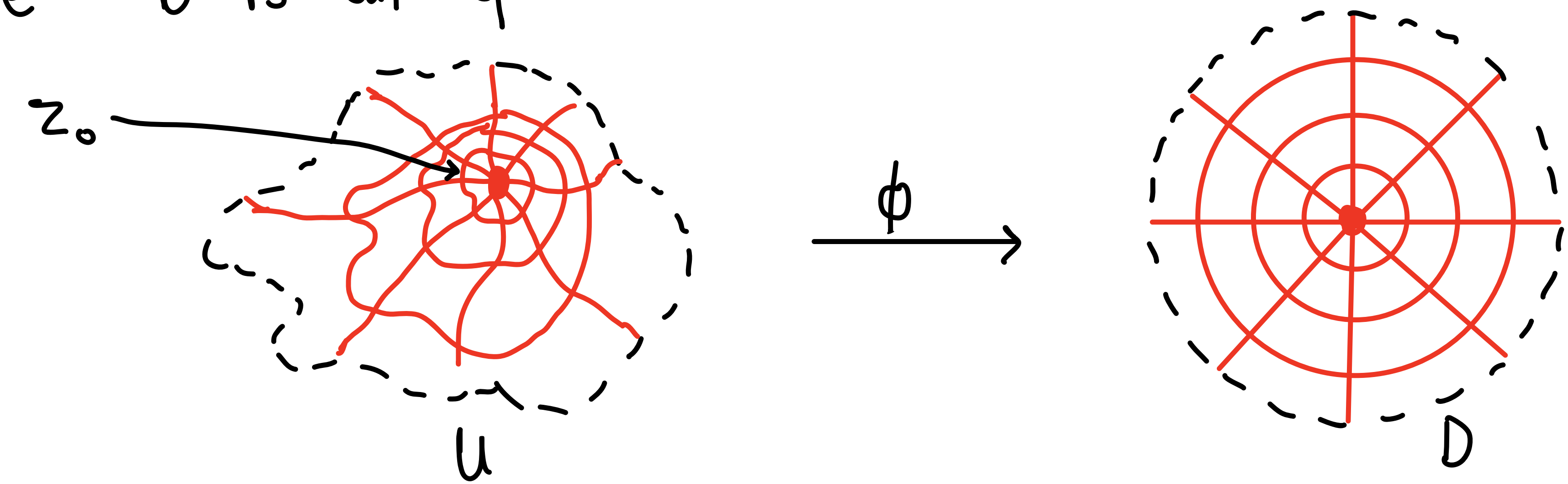
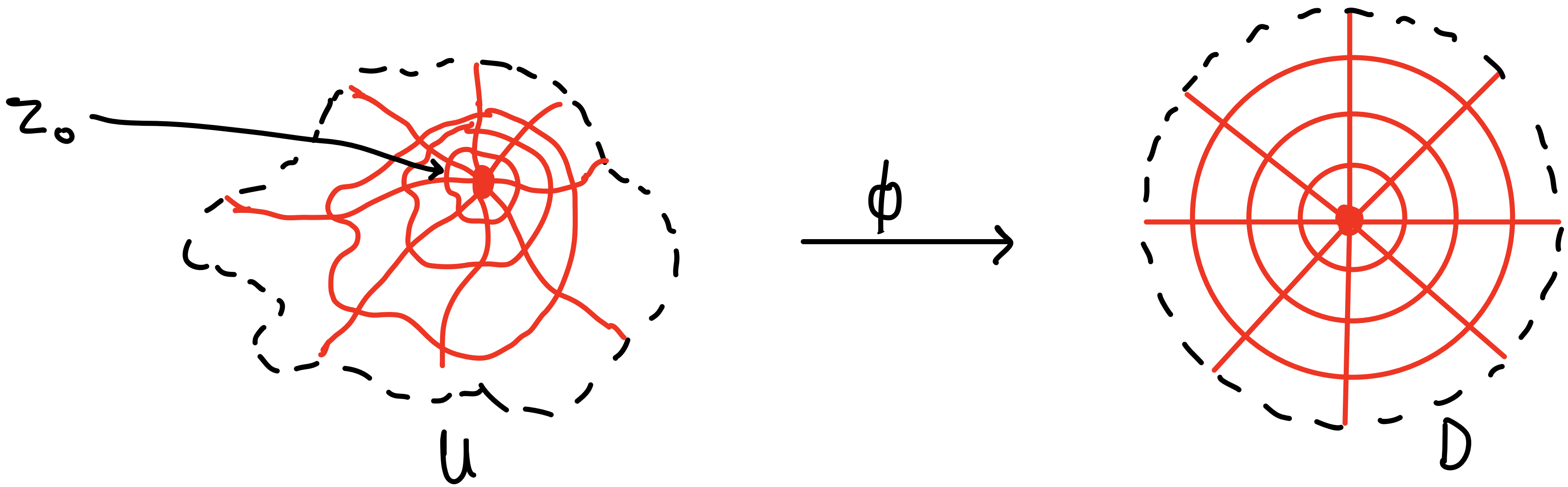



Pick $z_0 \in U$. Then in fact $\phi$ is unique if we demand $\phi(z_0) = 0$, $\phi'(z_0) = 1$.

<u>Question</u>: Can we write down an explicit formula for $\phi$?

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**Definition (Bergman 1950)** The **Bergman kernel** of a bounded domain $U \subseteq \mathbb{C}$ is

$$B(z,w) = \sum_{n=1}^{\infty} \overline{e_n(z)}\, e_n(w) \qquad (z, w \in U)$$


where $\{e_n\}_{n=1}^{\infty}$ is any orthonormal basis of holomorphic functions on $U$.

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**Example** For $U$ = unit dish, can take $e_n(z) = \sqrt{\frac{n+1}{\pi}}\, z^n$, $n = 0, 1, 2, \cdots$.

$$\therefore B(z,w) = \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1)\, \bar{z}^n w^n = \frac{1}{\pi(1-\bar{z}w)^2}$$

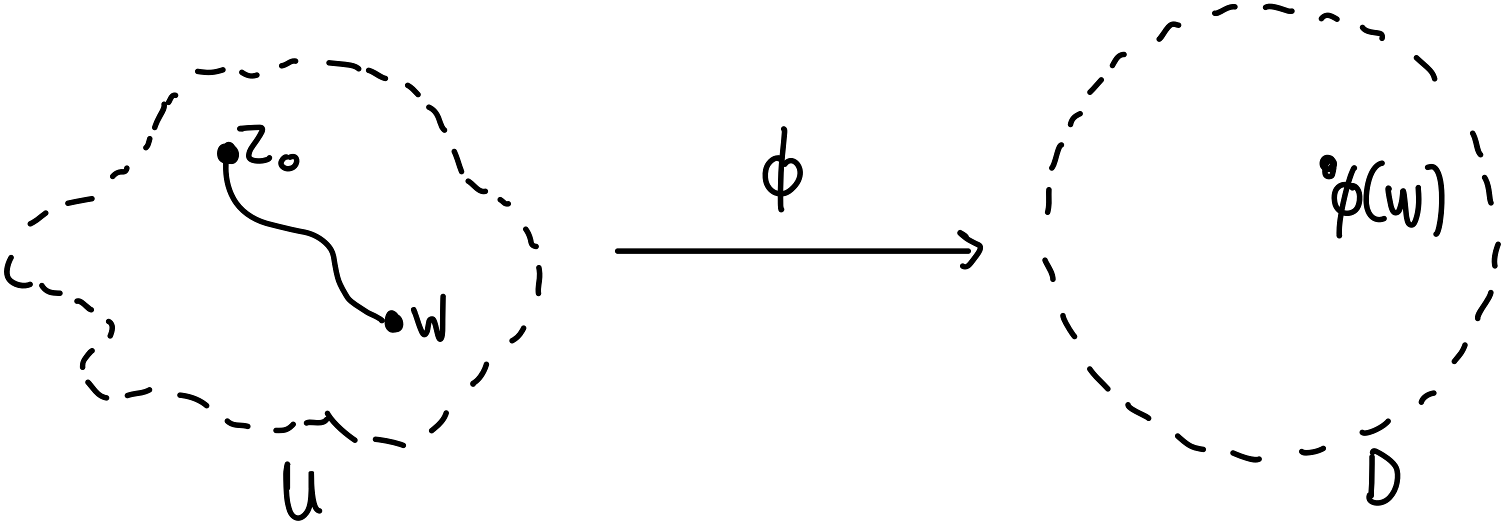
(the standard Poisson kernel)

**Theorem (Bergman 1950)** The Riemann map

$$\phi : U \longrightarrow D, \quad \phi(z_0) = 0, \quad \phi'(z_0) = 1$$

can be computed as follows:

$$\phi(w) = \frac{1}{B_U(z_0, z_0)} \int_{z_0}^{w} B_U(z_0, z) \, dz$$



From a different viewpoint, given the domain $U \subset \mathbb{C}$, define the <u>coherent state based at $z_0 \in U$</u> to be the holomorphic function

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It can be characterized as the holomorphic function on $U$ which is <u>maximally peaked at $z_0$</u>, i.e. for any holomorphic $f$ on $U$,

$$|f(z_0)|^2 \leq \psi_{z_0}(z_0) \iint_U |f|^2 \, dA$$

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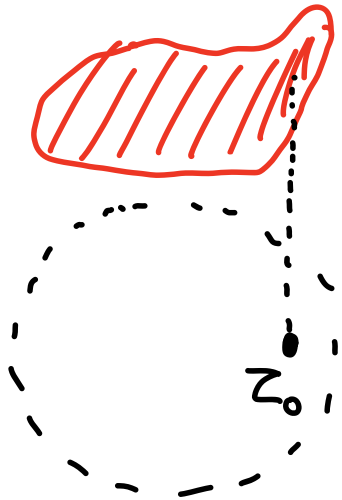


<u>Example</u> When $U$ = unit disk, $\psi_{z_0}(w) = \dfrac{1}{\pi(1-\bar{z}_0 w)^2}$

We can expand any $f \in \mathrm{Hol}(U)$ as a linear combination of the basis functions $\{e_n\}_{n=1}^{\infty}$:

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Similarly, and more geometrically, the coherent states $\{\psi_z\}_{z \in U}$ form an "continuous basis" in the sense that any $f \in \mathrm{Hol}(U)$ can be expanded as an <u>integral</u> linear combination of them:

$$f = \iint_{z \in U} \langle \psi_z, f \rangle \psi_z \, dA$$

## Take home message

- The Bergman kernel $B_U(z,w)$ of a bounded domain $U \subset \mathbb{C}$, and the associated coherent states, are fundamental geometric objects.

- The Riemann map

$$\phi : U \longrightarrow D \quad , \quad \phi(z_0) = 0, \quad \phi'(z_0) = 1$$

can be explicitly expressed in terms of the coherent state $\psi_{z_0}$.

- Every $f \in Hol(U)$ can be expressed as an integral linear combination of the coherent states. The coefficient of $\psi_z$ is just $f(z)$!

## 2. Geometric generalization

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bounded domain $\rightsquigarrow$
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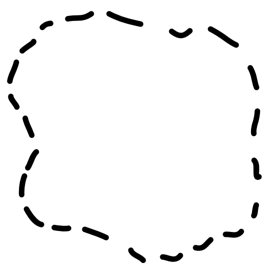
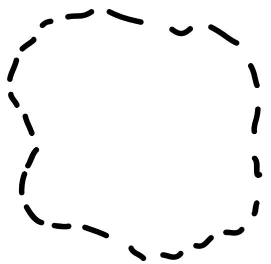


$U$

# 2. Geometric generalization

We now generalize:

bounded domain $U \subset \mathbb{C}$ $\rightsquigarrow$ compact complex manifold $X$



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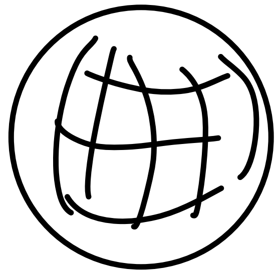
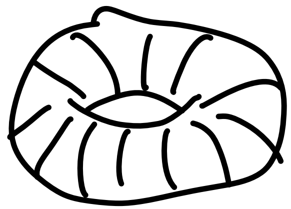
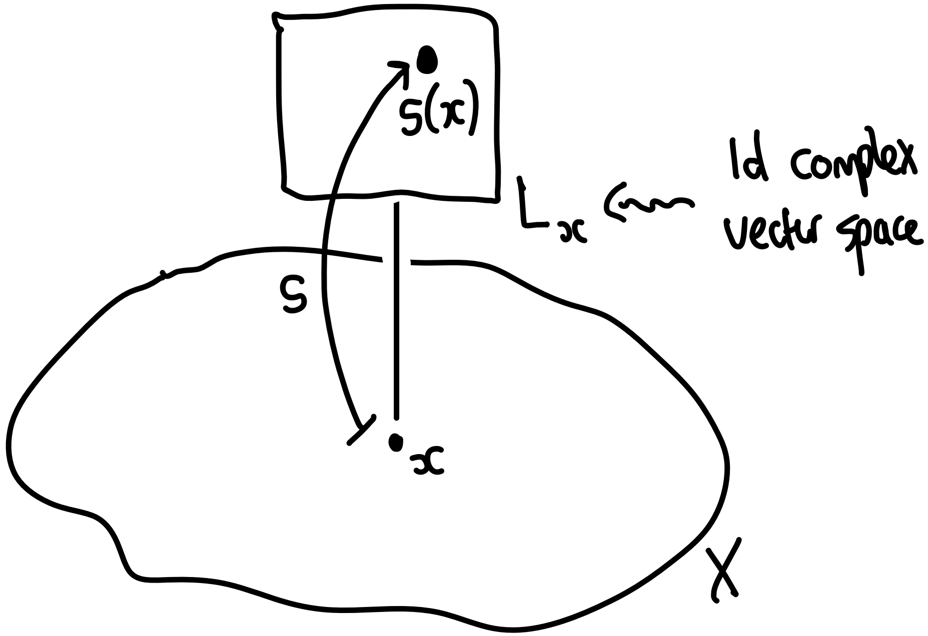
eg.  , 

$S^2$ , elliptic curve

$\mathbb{CP}^n$, smooth projective variety holomorphic function $f$ on $U$ $\rightsquigarrow$

holomorphic function $f$ on $U$ $\rightsquigarrow$ holomorphic section $s$ of a fixed holomorphic line bundle $L$ on $X$

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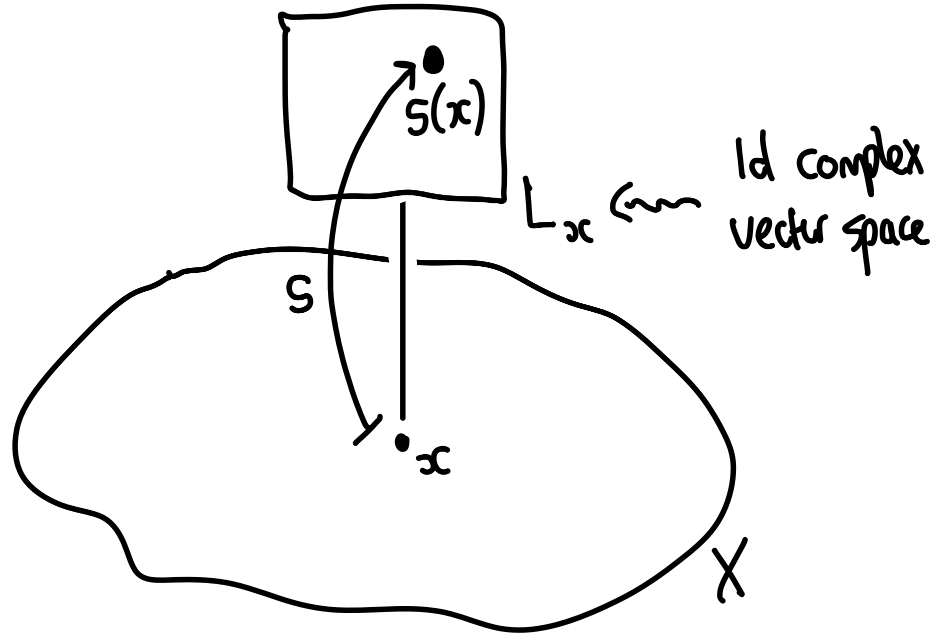
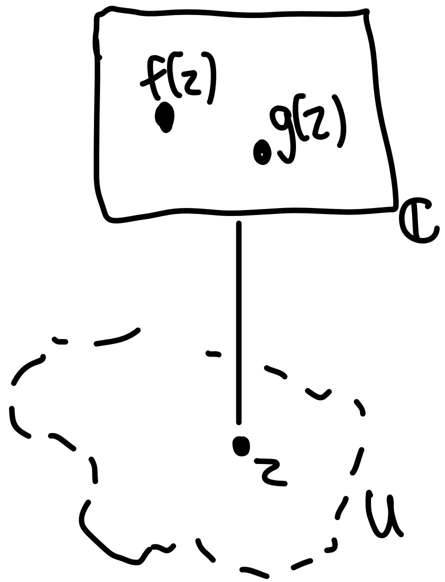



Cool fact: The vector space $\mathrm{Hol}(X,L)$ of holomorphic sections of $L$ is finite-dimensional!

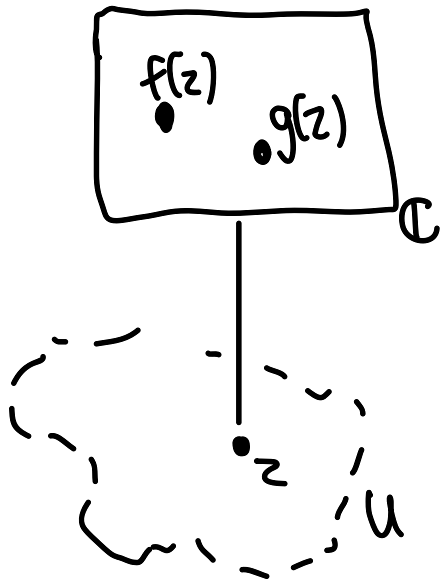
$$\langle f, g \rangle := \iint_{z \in U} \bar{f}(z) g(z) \, dA \quad \rightsquigarrow$$

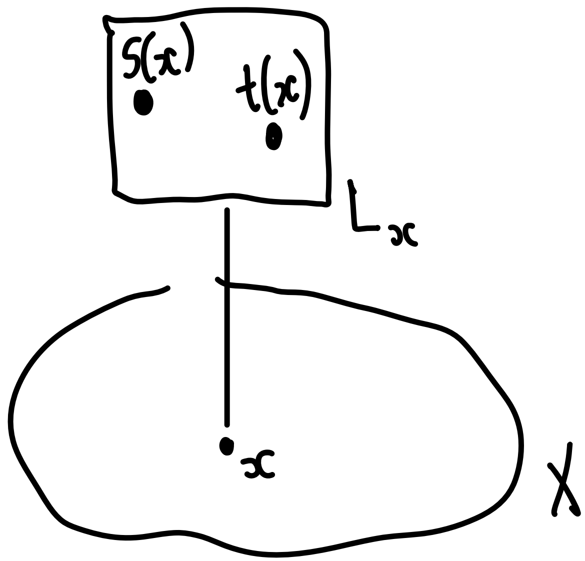


$$\langle f,g\rangle := \iint_{z\in U} \bar{f}(z)g(z)\,dA \rightsquigarrow \langle s,t\rangle := \int_{x\in X} \big(s(x),t(x)\big)_x \,\mathrm{vol}_x$$

where we have fixed:

- a smooth choice of an inner product $(\cdot,\cdot)_x$ on each $L_x$
- a volume form on $X$



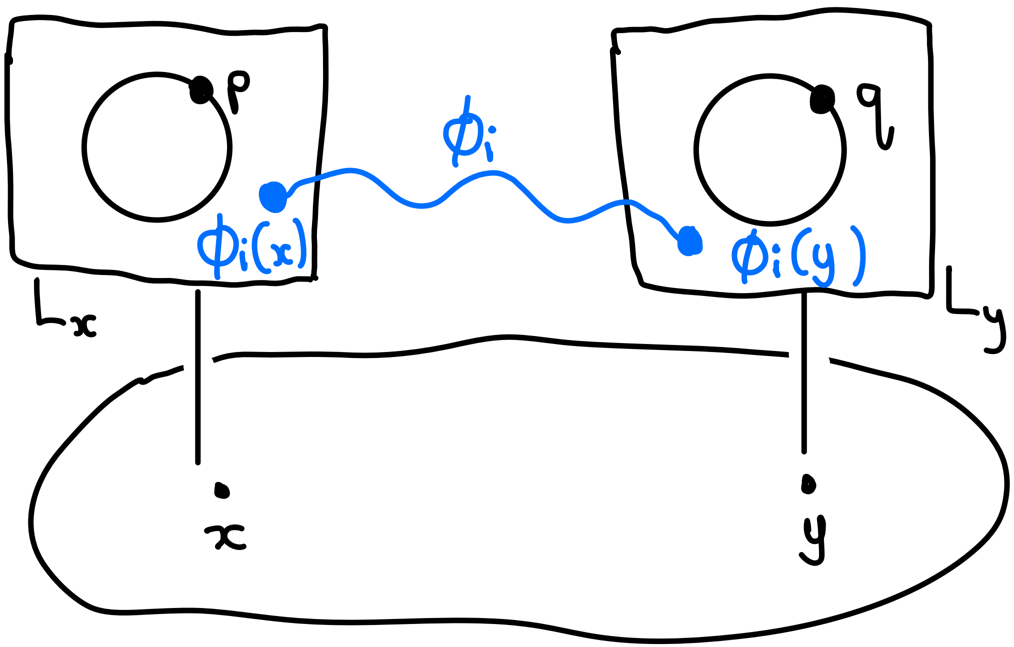
**Definition** The **Bergman kernel** of the data $(X, L, (\cdot,\cdot), \mathrm{vol})$ is

$$B : P \times P \longrightarrow \mathbb{C}$$

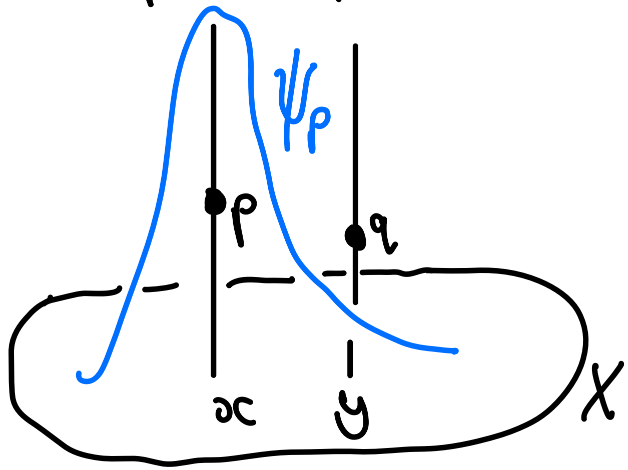
$$(p,q) \longmapsto \sum_{i=1}^{n} \overline{\phi_i(p)}\, \phi_i(q)$$

the unit circle bundle of $L^\vee$

where $x = \pi(p)$, $y = \pi(q)$ and $\{\phi_i\}_{i=1}^n$ is an orthonormal basis for $\mathrm{Hol}(X,L)$.



Similarly, the <u>coherent state</u> $\psi_p$ based at $x \in X$ with phase $p \in P_x \in L_x^\vee$ is the holomorphic section of $L$ defined by:




$$\psi_p(q) = B(p,q)$$

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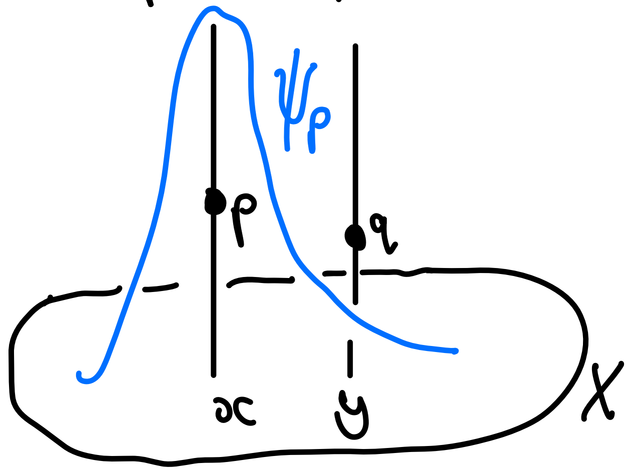


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Alternatively, consider the evaluation linear functional

$$\begin{aligned} ev_p : \mathrm{Hol}(X,L) &\longrightarrow \mathbb{C} \\ s &\longmapsto (p, s(x))_x \end{aligned}$$

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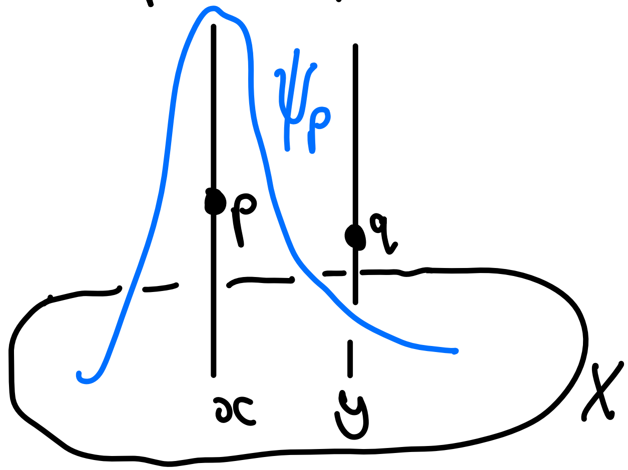


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Then $\psi_p \in \mathrm{Hol}(X, L)$ is the vector representing $ev_p$, i.e.

$$\langle \psi_p, s \rangle = ev_p(s) \quad \text{for all } s \in \mathrm{Hol}(X, L).$$

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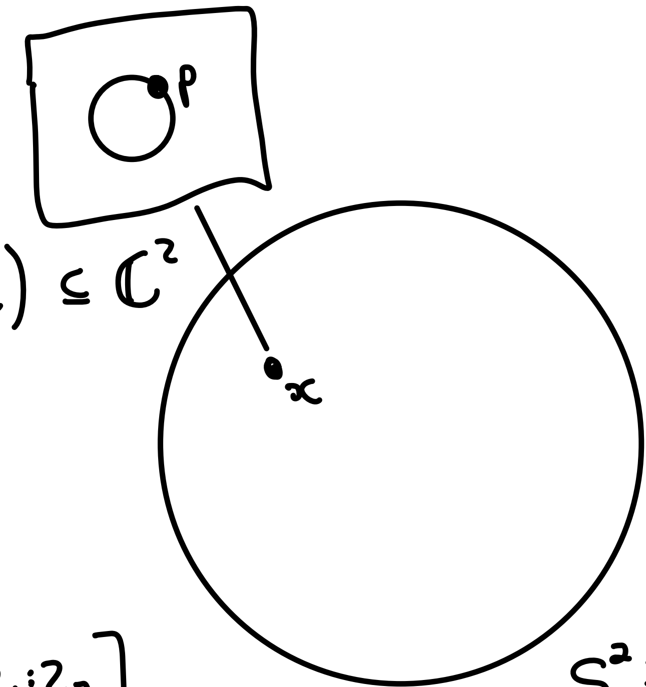


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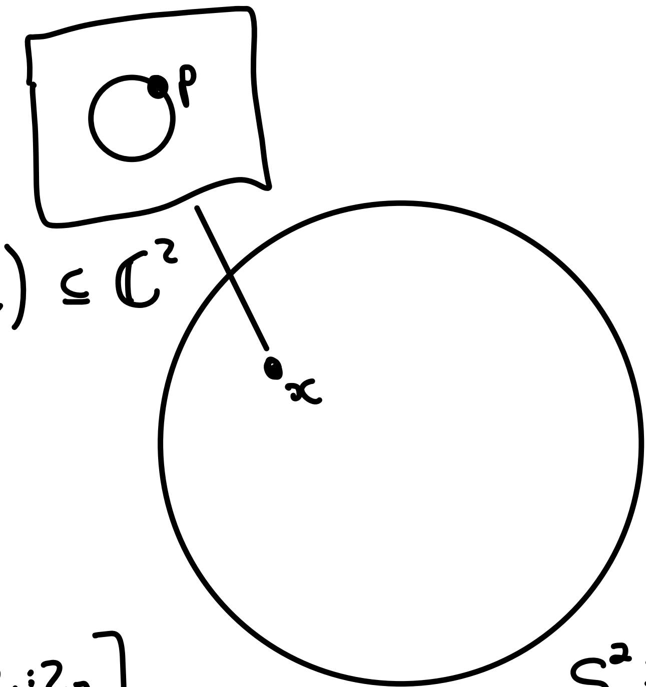


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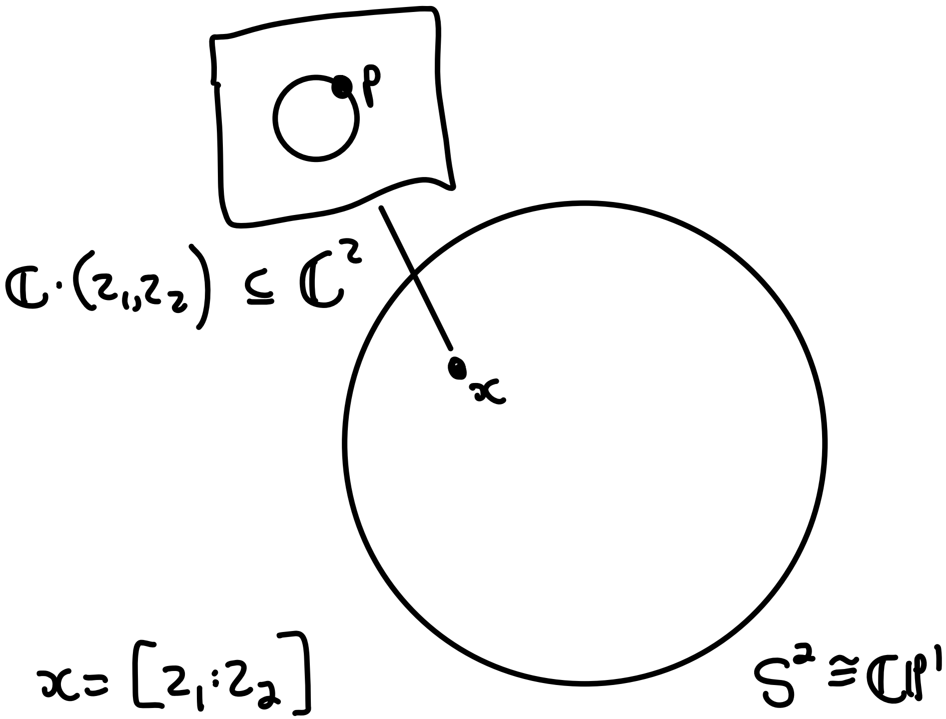


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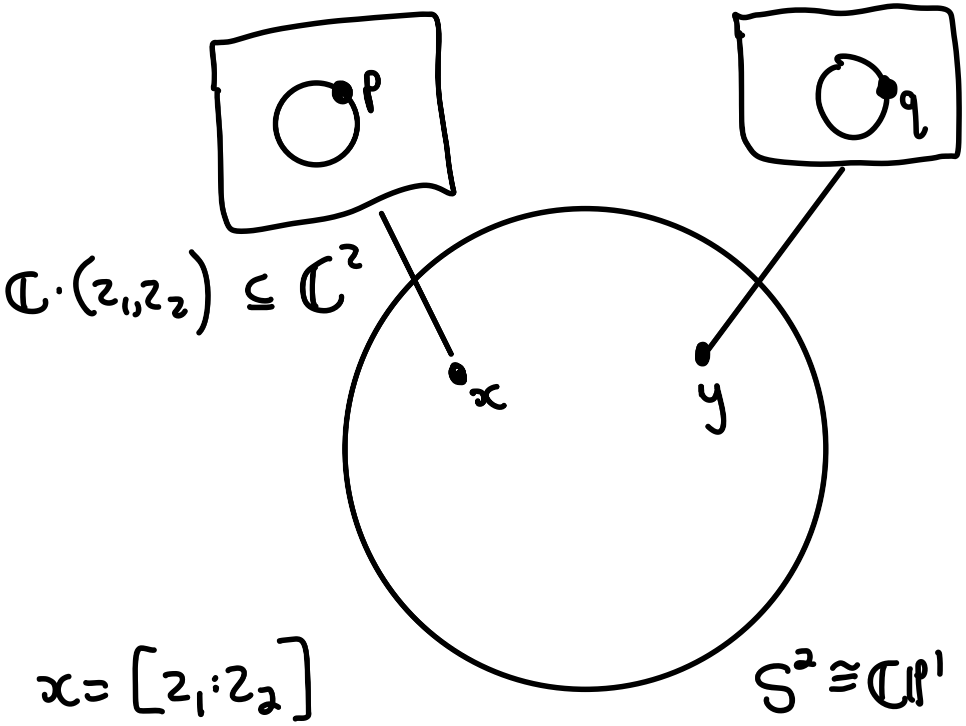
$$\begin{aligned} \pi : S^3 &\longrightarrow S^2 \\ (z_1,z_2) &\longmapsto [z_1 : z_2] \end{aligned}$$

is the Hopf fibration!

**Theorem** (B-N) For $X = S^2$ $(= \mathbb{CP}^1)$ and $L_k = (\tau^\vee)^{\otimes k}$, the Bergman kernel computes as:

$$B_k(p,q) = \frac{k+1}{2\pi} \langle p, q \rangle_{\mathbb{C}^2}^k$$

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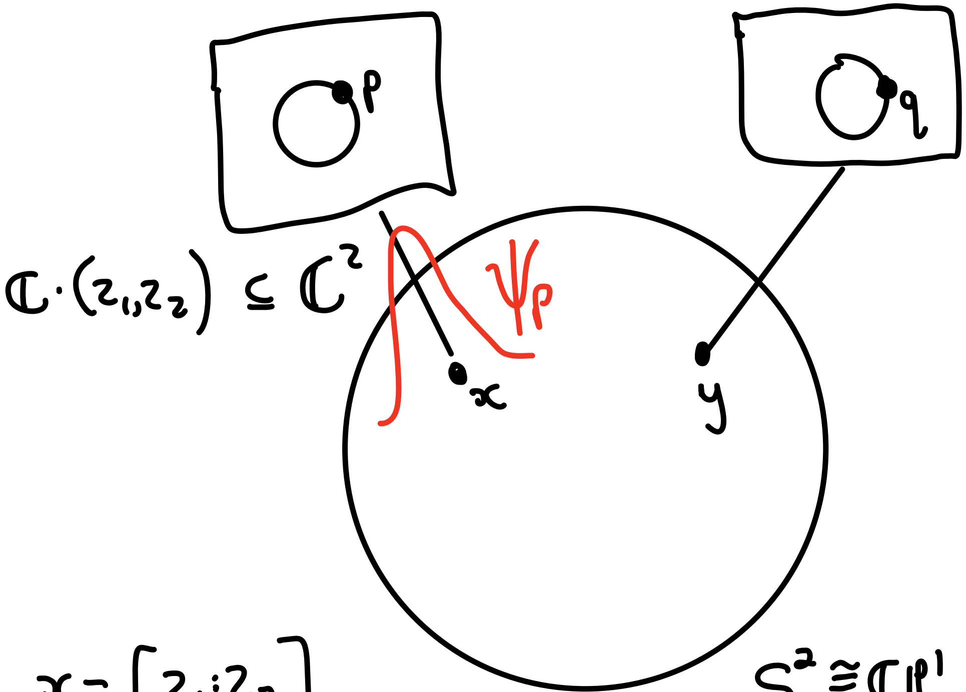


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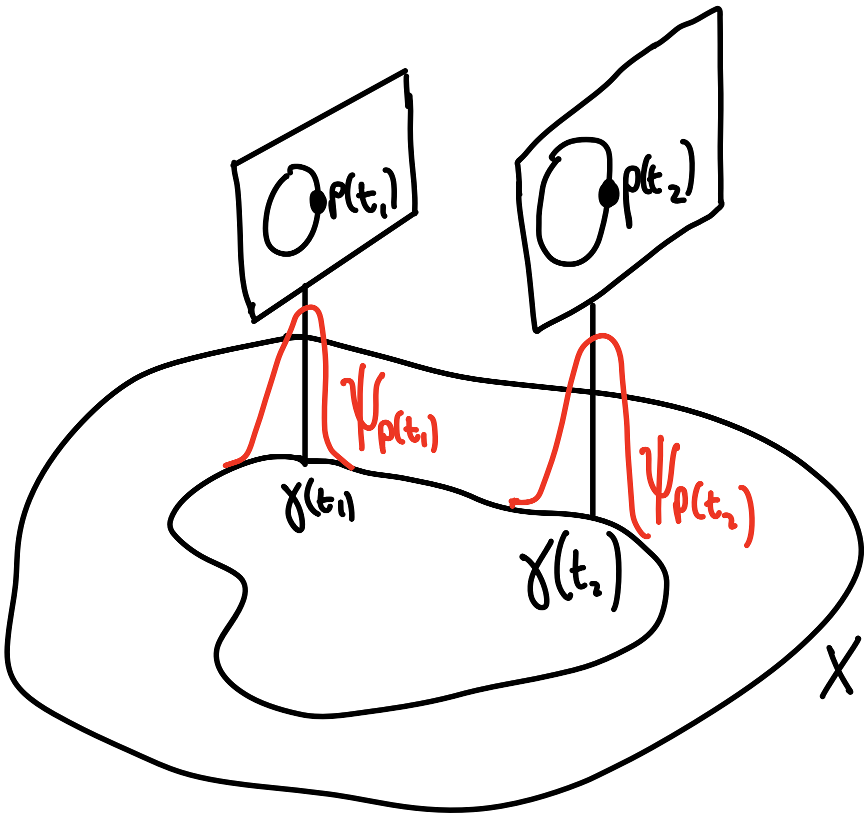


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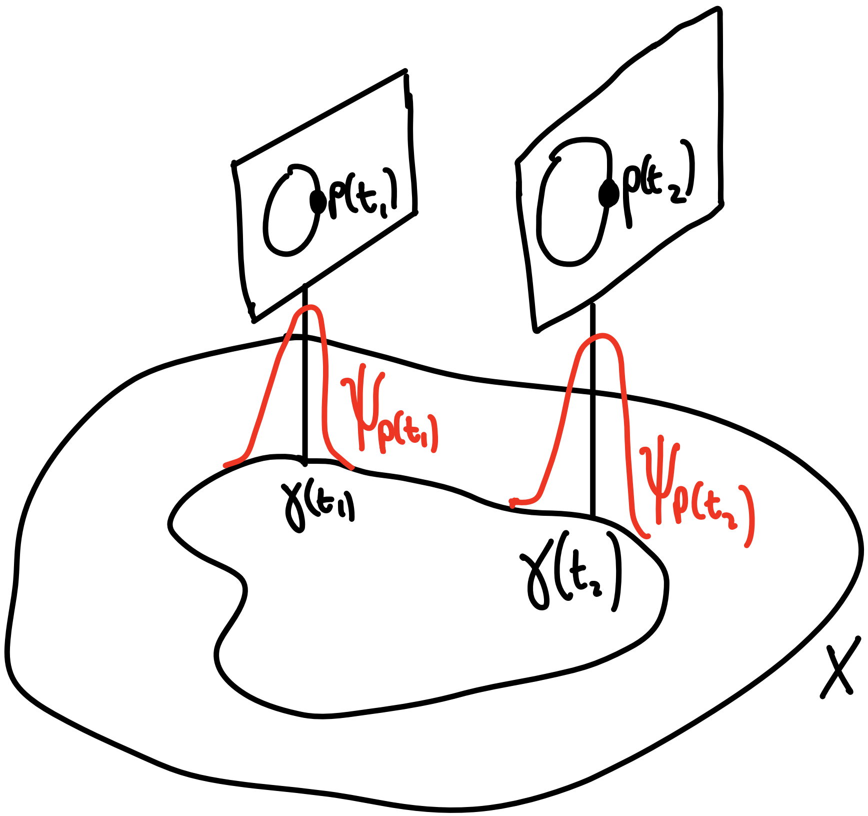


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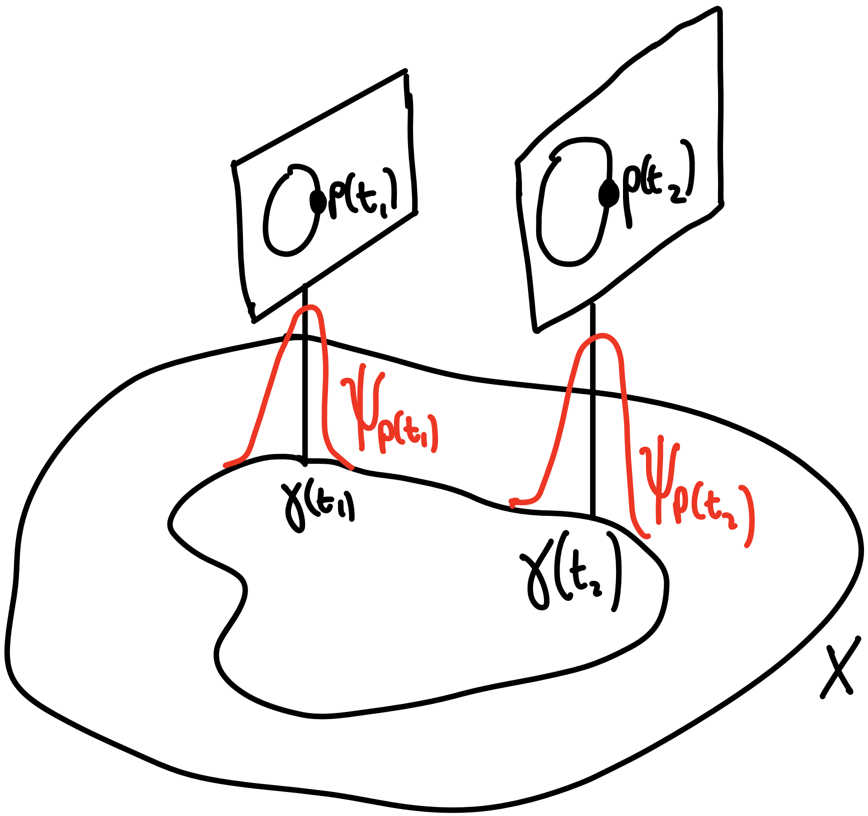


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A parallel transported lift of $\gamma$ to $P$

$\therefore$ Need parallel transport around $\gamma$ to be the identity!

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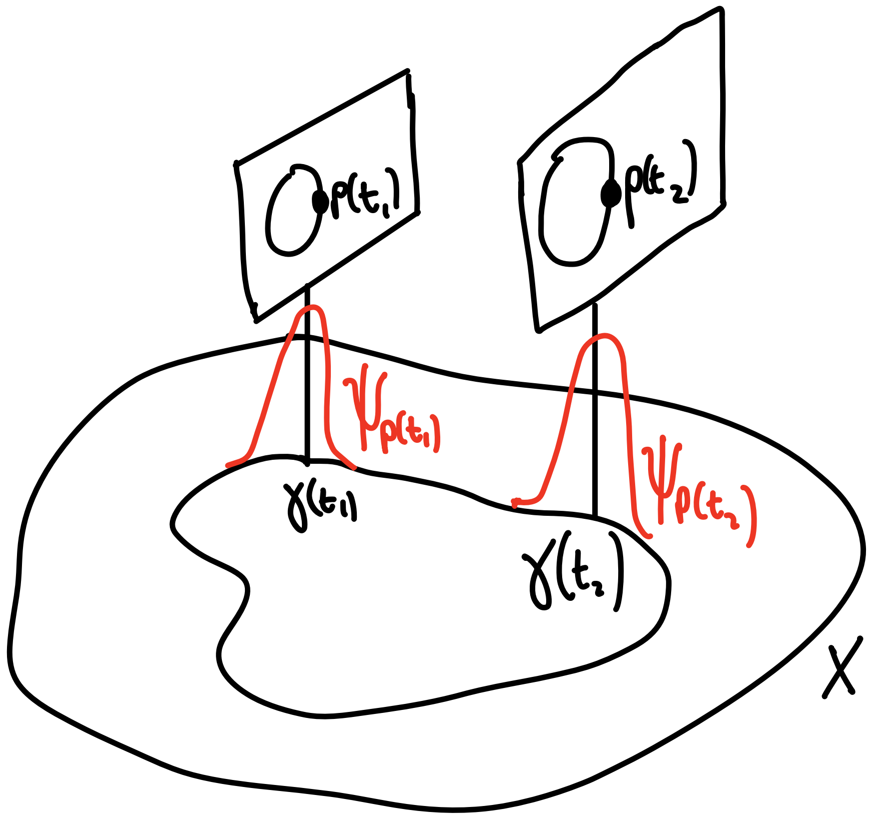


**Defn** (B-N)

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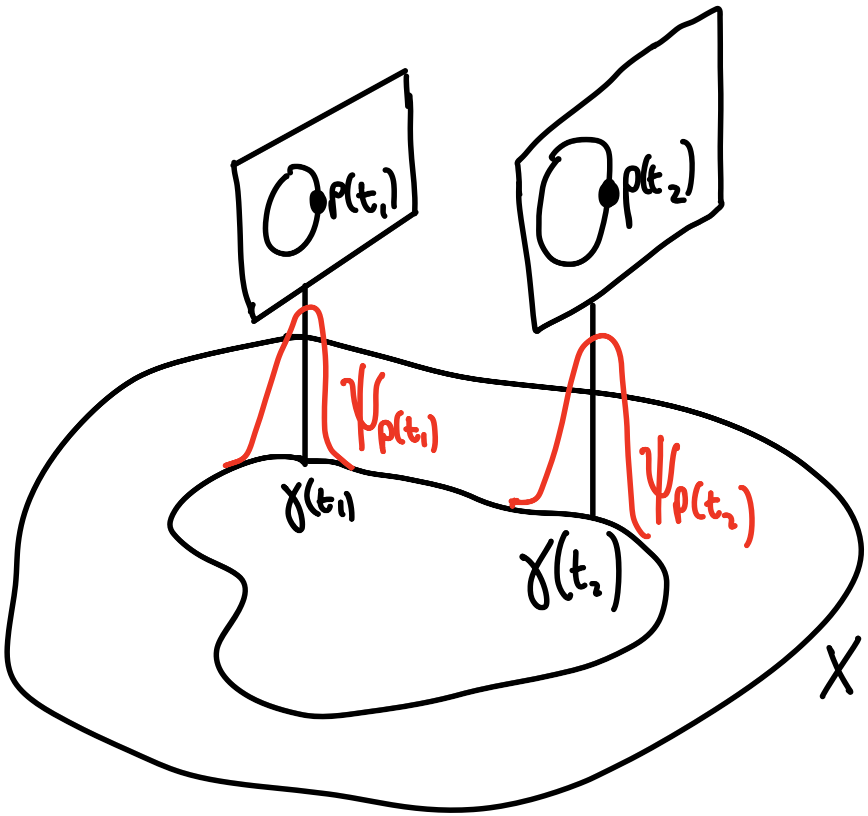


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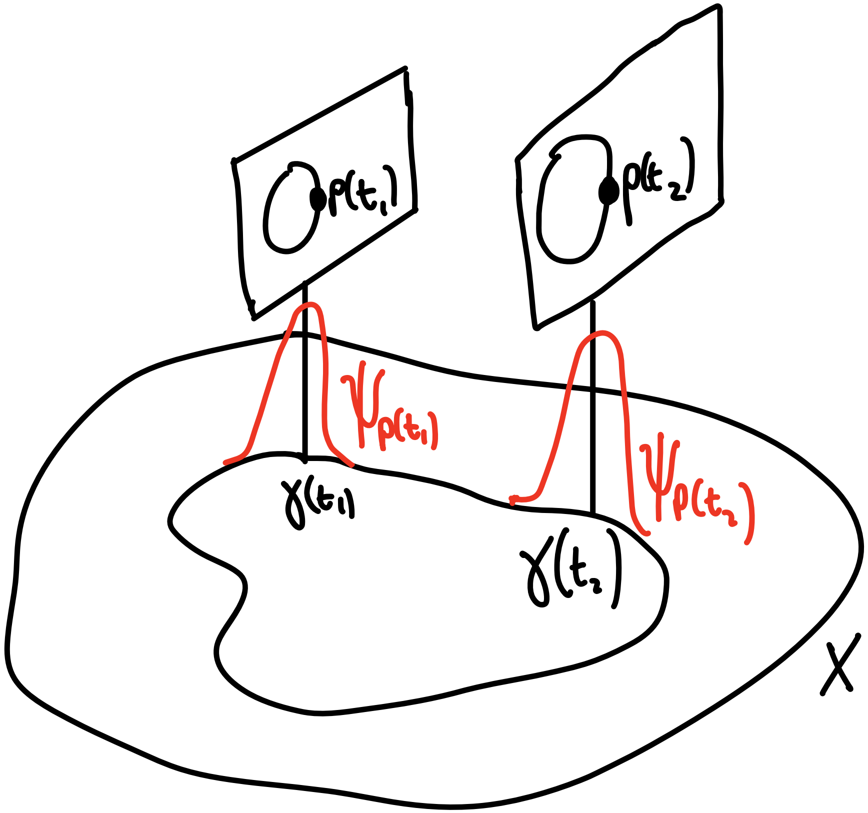


<u>Defn</u> (B-N) Suppose parallel transport in $P$ around $\gamma$ is a $k^{th}$ root of unity.

i.e. $\gamma$ is a <u>Bohr-Sommerfeld loop at level $k$</u> in language of Borthwick, Paul and Uribe

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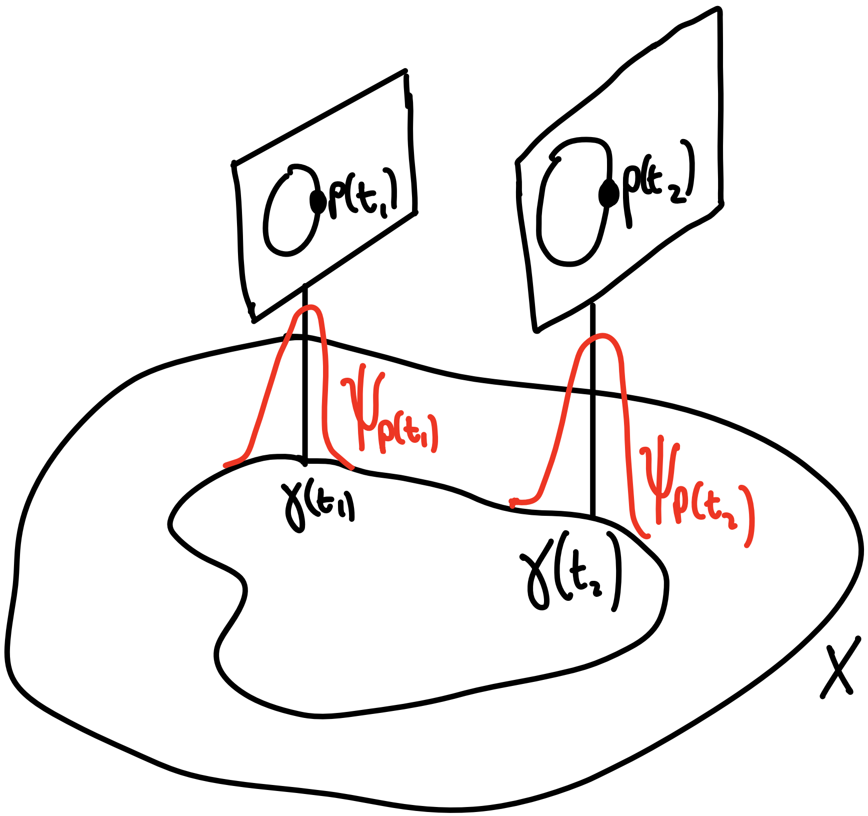
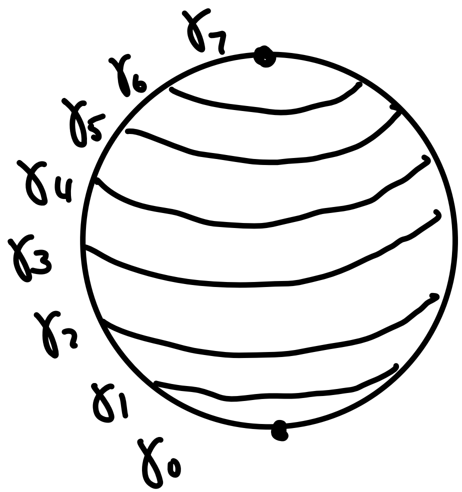



**Defn** (B-N) Suppose parallel transport in $P$ around $\gamma$ is a $k^{th}$ root of unity. The <u>coherent loop state at level $k$</u> is

$$\Psi_\gamma^{(k)} := \int_{t=0}^{T} \psi_{p(t)}^{(k)} \, dt \quad \in \mathrm{Hol}(X, L^{\otimes k})$$

**Theorem (B-N)** A loop $\gamma$ at height $z$ on $S^2$ is a Bohr-Sommerfeld loop at level $k$ if and only if $z = 1 - \frac{2n}{k}$, $n = 0, 1, 2, \cdots, k$.

eg. $k=7$



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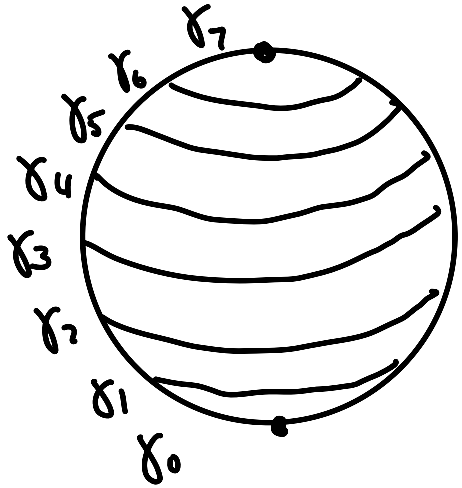


**Theorem (B-N)** The corresponding coherent loop states are precisely the angular momentum eigenstates $|j,m\rangle$ in quantum mechanics, i.e.

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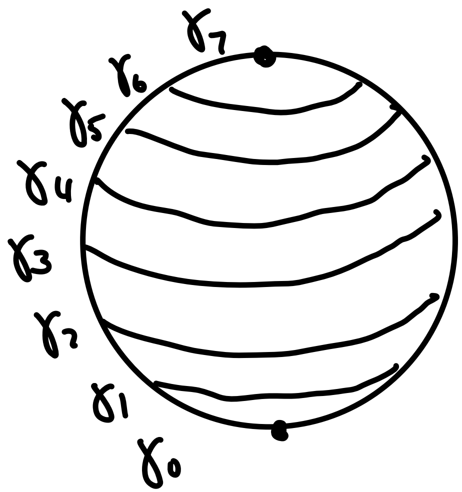


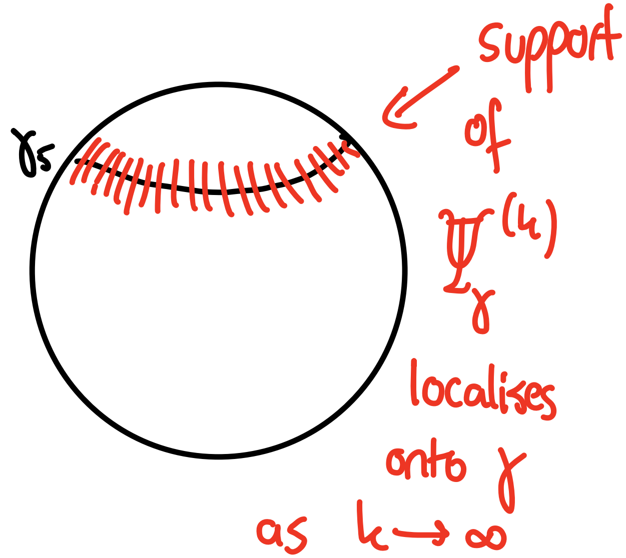


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$$V_k = \mathrm{Hol}\left( S^2, (\mathcal{T}^{\vee})^{\otimes k} \right)$$

This is a geometric description of the unique irreducible representation of $SU(2)$ of dimension $k+1$.

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$$d^j_{mm'}(\beta) := \langle j, m' \mid U_y(\beta) \mid j, m \rangle$$

Littlejohn and Yu wrote down a beautiful asymptotic formula for these matrix elements in a physics-style paper (i.e. no formal proof):

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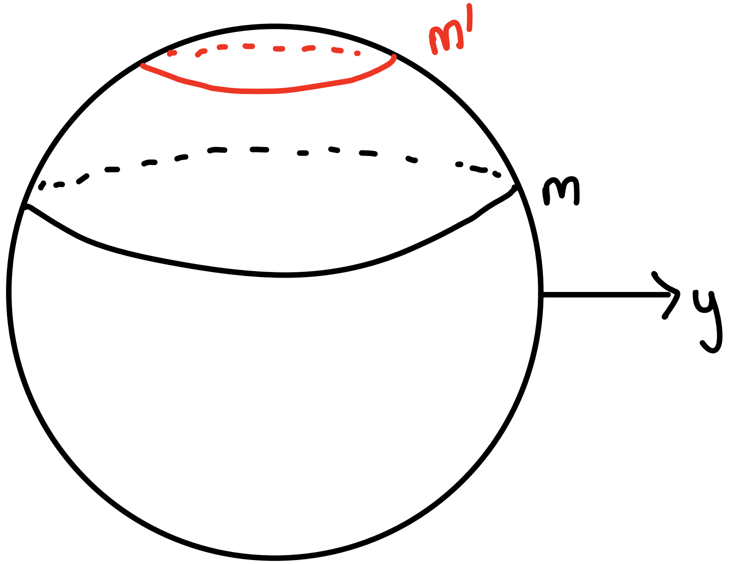

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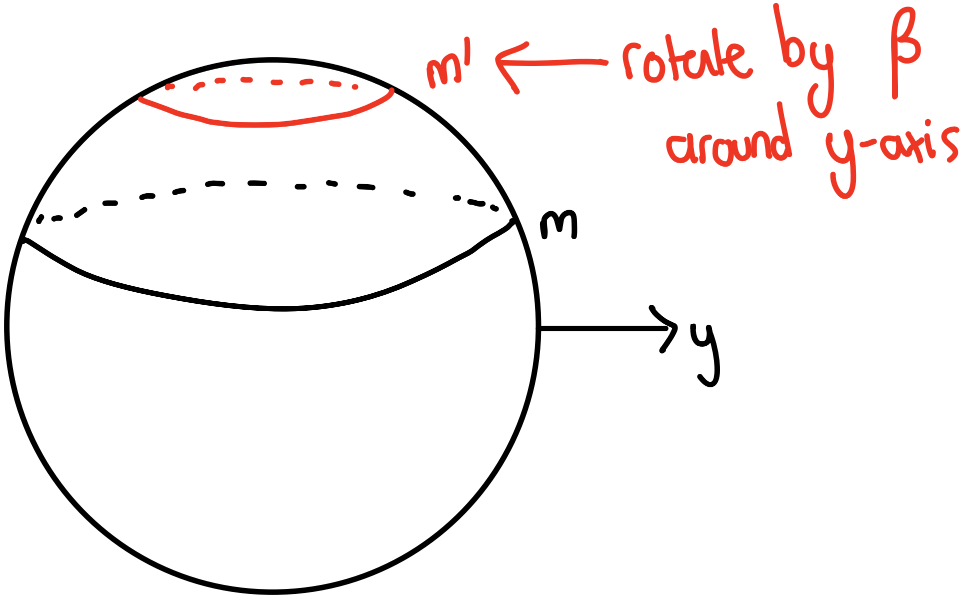

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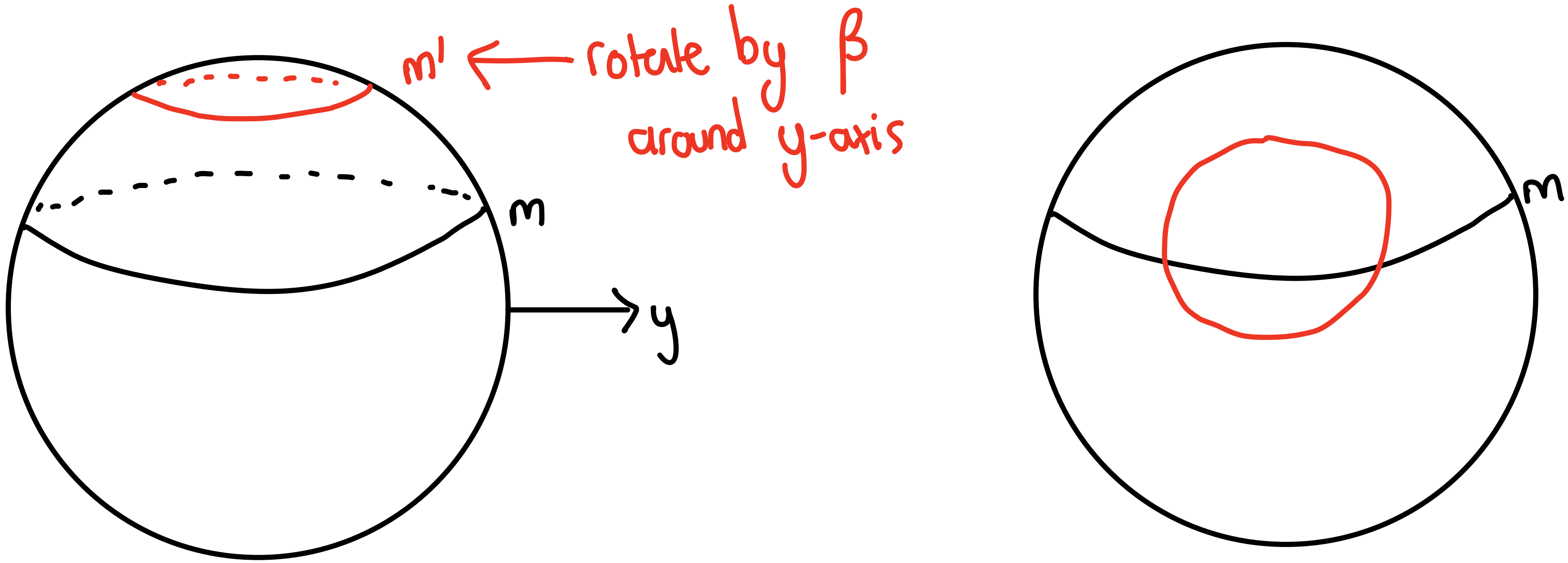

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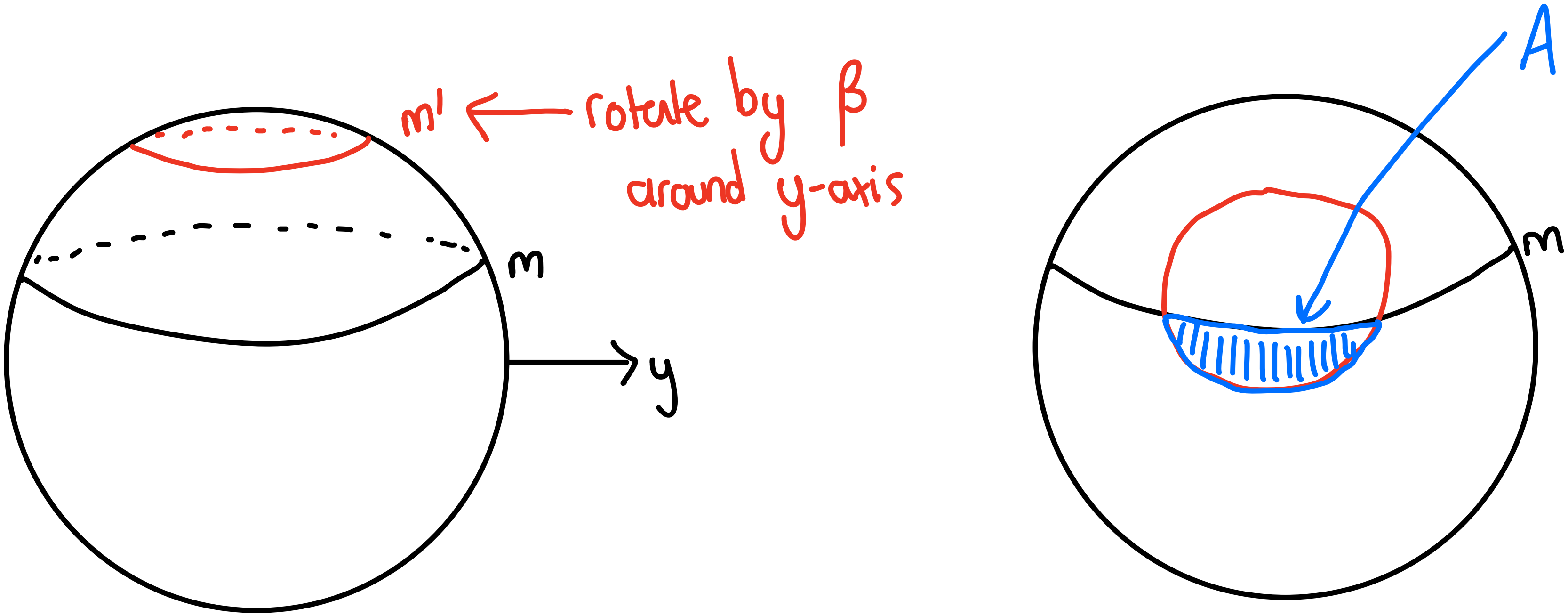

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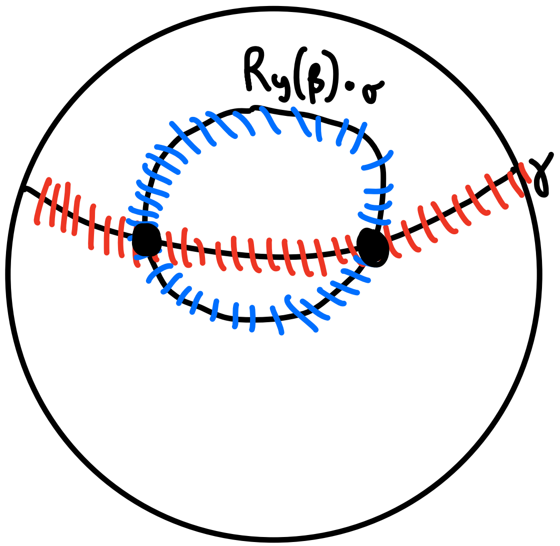

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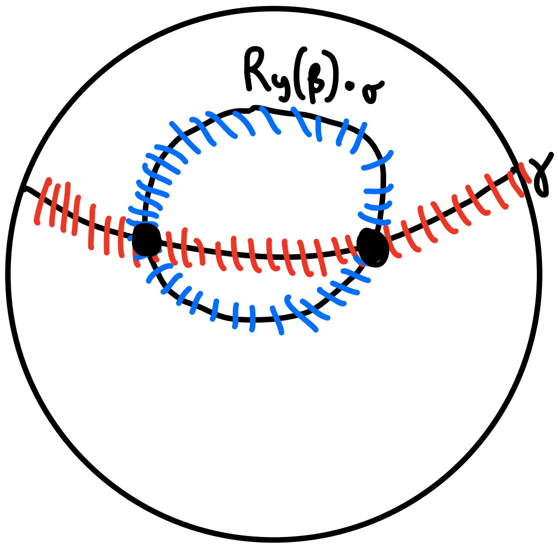

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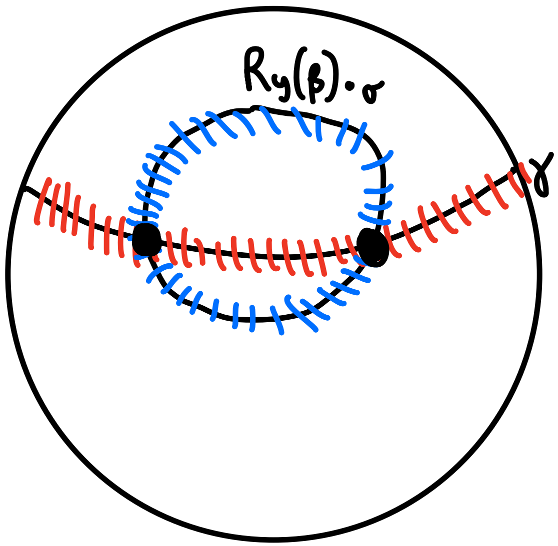


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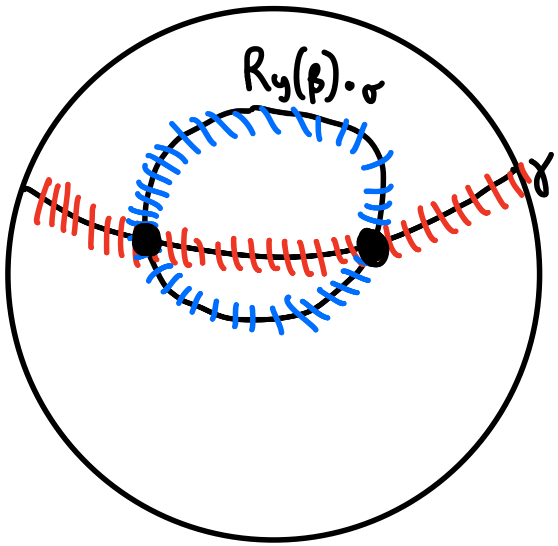


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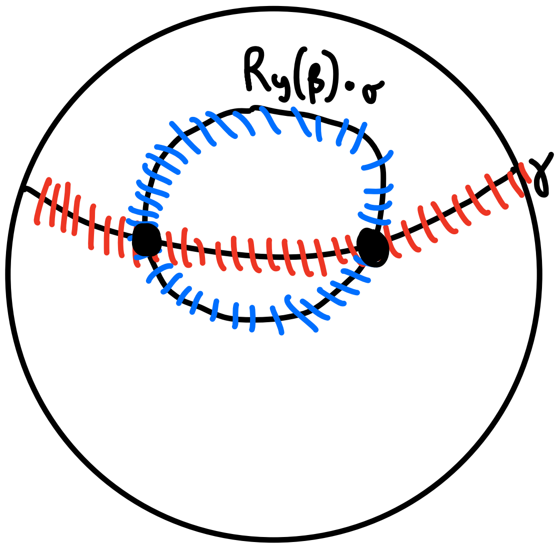


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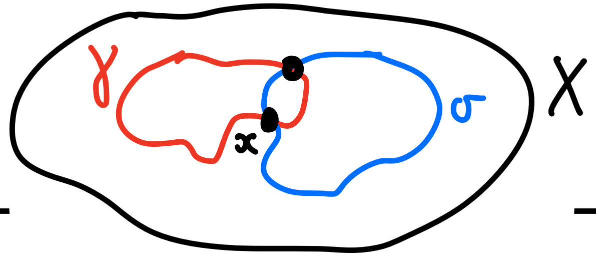
= Littlejohn and Yu's formula! $\square$

In fact, Borthwick, Paul and Uribe established a general formula for the asymptotics of the inner product of two coherent loop states for <u>any</u> line bundle over a Kähler manifold:




<u>Theorem (BPU)</u> As $k \longrightarrow \infty$,

$$\langle \Psi_\gamma , \Psi_\sigma \rangle \sim \sqrt{2} \sum_{x \in \gamma_1 \cap \gamma_2} \frac{\omega_x^k \, e^{-i(\theta_x/2 \, - \pi/4)}}{\sqrt{\sin \theta_x}}$$

<u>Proof</u> (BPU) Complicated! Uses result of Boutet de Monvel and Guillemin. $\square$

<u>Proof</u> (B-N) Loading... just uses stationary phase and elementary Kähler geometry. $\square$

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- Poincaré series (modular forms for $\Gamma \subset SL_2\mathbb{R}$) are also coherent loop states!