Interior operators and the category of (pre)sheaves

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Motivation

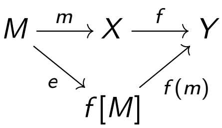
- Motivated by the theory of categorical closure operators, the categorical notion of interior operators was introduced by [Vorster, 2000]. Unlike the case of categorical closure operators, the continuity condition of categorical interior operators can not be described in terms of images.
- While categorical closure and interior operators characterize each other in a category with a categorical transformation operator (see [Vorster, 2000]), the two operators are not categorically dual to each other, that is: it is not true in general that whatever one does with respect to closure operators may be done relative to interior operators and vice versa. In particular, in the category of groups the two notions are not equivalent.
- As a consequence, the study of categorical interior operators in their own right is interesting.

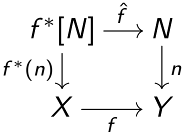
Background

- Consider a finitely complete category $\mathbb{C}$ with a proper $(\mathcal{E}, \mathcal{M})$-factorization system for morphisms.
- The category $\mathbb{C}$ is further assumed to be $\mathcal{M}$-complete so that arbitrary $\mathcal{M}$-intersections of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$.
- For each object $X \in \mathbb{C}$, sub$X$ denotes the class of $\mathcal{M}$-morphisms with codomain X and its objects are known as ($\mathcal{M}$-)subobjects of X. This class is preordered with the relation $m \leq n \Leftrightarrow \ m = n \circ j$ for some j (necessarily in $\mathcal{M}$);

Background

- For each $X \in \mathbb{C}$, sub$X$ is a complete lattice with $0_X : O_X \to X$ and $1_X : X \to X$ as the least and greatest member of the lattice, respectively. The meet of $s : S \to X$ and $r : R \to X$ in sub$X$ is given by $s \wedge r \cong s \circ s^*(r) \cong r \circ r^*(s)$.
- Given a morphism $f : X \to Y$ in $\mathbb{C}$, $m \in$ sub$X$ and $n \in$ sub$Y$ we define the image $f(m)$ in sub$X$ of m under f as the $\mathcal{M}$-component of the $(\mathcal{E}, \mathcal{M})$-factorization of $f \circ m$, preimage $f^*(n)$ in sub$Y$ of n under f as the pullback of n along f. By the image of f we mean $f(1_X)$.





Background

Remark

Let $f : X \to Y$ be a morphism in $\mathbb{C}$, $m \in \text{sub}X$ and $n \in \text{sub}Y$. Then f induces image-preimage Galois connection

$$\text{sub}X \underset{f^*(-)}{\overset{f(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \text{sub}Y .$$ Consequently, $f^*(1_Y) \cong 1_X$, $f(0_X) \cong 0_Y$. Moreover, if $f \in \mathcal{M}$ then $f \circ m \cong f(m)$

We assume that the preimage $f^*(-) : \text{sub}Y \to \text{sub}X$ preserves arbitrary joins for every morphism $f : X \to Y$ in $\mathbb{C}$. Consequently, f^* has a right adjoint f_*, which is given by $f_*(r) = \bigvee\{u \in \text{sub}Y : f^*(u) \leq r\}$. Hence, one has $f^*(k) \leq r$ if and only if $k \leq f_*(r)$ for all $r \in \text{sub}X$ and $k \in \text{sub}Y$. Note that the above assumption fails in algebraic categories in general. That is, pullbacks may not preserve the joins of subobjects.

Background

The above assumption implies:

Proposition

Let $f : X \to Y$ in $\mathbb{C}$, $r \in \mathrm{sub}X$ and $n \in \mathrm{sub}Y$. Then: f induces a preimage-dual image Galois connection:

$$\mathrm{sub}Y \underset{f_*(-)}{\overset{f^*(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{sub}X$$

Consequently, $f^*(0_Y) \cong 0_X$, $f_*(1_X) \cong 1_Y$.

Remark

For each $X \in \mathbb{C}$, sub$X$ has a structure of a frame. Indeed, as is mentioned before each sub$X$ is a complete lattice. Moreover, one has $m \wedge \bigvee_{i \in I} r_i \cong m \circ m^* \left(\bigvee_{i \in I} r_i \right) \cong m \circ \bigvee_{i \in I} m^*(r_i) \cong m \left(\bigvee_{i \in I} m^*(r_i) \right) \cong \bigvee_{i \in I} m(m^*(r_i)) \cong \bigvee_{i \in I} m \circ m^*(r_i) \cong \bigvee_{i \in I} (m \wedge r_i)$ for all $m, r_i \in$ sub$X$, $i \in I$, hence meets distribute over arbitary joins in each sub$X$.

In the sequel we use $\neg m$ to denote the pseudocomplement of $m \in$ sub$X$.

Interior operators

We make use of the following definition from [Vorster, 2000].

Definition

An interior operator i on $\mathbb{C}$ with respect to $\mathcal{M}$ is a family

$$i = (i_X : \mathrm{sub}X \to \mathrm{sub}X)_{X \in \mathbb{C}}$$

of functions which are

(I_1) contractive: $i_X(m) \leq m$,

(I_2) order preserving: if $k \leq m$ then $i_X(k) \leq i_X(m)$,

(I_3) and which satisfy the continuity condition:
$f^*(i_Y(n)) \leq i_X(f^*(n))$,

for all $f : X \to Y$ in $\mathbb{C}$ and $k, m \in \mathrm{sub}X$ and $n \in \mathrm{sub}Y$.

Interior operators

The following terminologies will be used whenever necessary.

Definition

We call

(*a*) an $\mathcal{M}$-subobject $m : M \to X$ i-open (in X) if $i_X(m) \cong m$;

(*b*) i idempotent if $i_X(i_X(m)) \cong i_X(m)$ for all $m \in \text{sub}X$, $X \in \mathbb{C}$;

(*c*) i standard if $i_X(1_X) \cong 1_X$ for all $X \in \mathbb{C}$.

Interior operators

Now we provide some examples of interior operators.

Examples

(*a*) Let $\mathbb{C} = \textbf{Top}$ with (*Surjections*, *Embeddings*)-factorization system and $M \subseteq X \in \textbf{Top}$. The assignments

(*i*) $k_X^{\text{in}}(M) = \bigcup\{O \text{ open in } X : O \subseteq M\}$,

(*ii*) $k_X^{*\text{in}}(M) = \bigcup\{C \text{ closed in } X : C \subseteq M\}$, and

(*iii*) $q_X^{\text{in}}(M) = \bigcup\{O \text{ clopen in } X : O \subseteq M\}$

define standard and idempotent interior operators.

(*b*) Let $\mathbb{C} = \textbf{Grp}$ with the *(Surjective homorphisms, Injective homomorphisms)*-factorization system and $H \leq G \in \textbf{Grp}$. $n_G(H) = \bigvee\{N \trianglelefteq G : N \leq H\}$ defines a standard, and idempotent interior operator.

(*c*) Consider $\mathbb{C} = \textbf{Rng}$ with the *(Surjective homorphisms, Injective homomorphisms)*-factorization system and $S \leq R \in \textbf{Rng}$. Then the assignment $j_R(S) = \bigvee\{I \trianglelefteq R : I \leq S\}$ is a standard, and idempotent interior operator.

Interior operator on $\mathbb{C}$

Proposition

The assignment $i^{\neg}$ that to each subobject m of an object X in the category associates $i^{\neg}_X(m) = \bigvee\{n \in \mathrm{sub}X \mid \neg n \vee m \cong 1_X\}$ is a standard interior operator on the category. Note that the $i^{\neg}$-open subobjects of an object X are precisely the complemented ones. Clearly, if each subobjects are Boolean algebras, then $i^{\neg}$ coincides with the discrete interior operator.

Proof.

1. The contraction and monotonicity conditions of i are straightforwad.
2. Let $f : X \to Y$ in $(\mathbb{C})$ and $k \in \mathrm{sub}Y$. Since $\neg n \vee k \cong 1_Y$ implies
$1_X \cong f^*(1_Y) \cong f^*(\neg n \vee k) \cong f^*(\neg n) \vee f^*(k) \leq \neg f^*(n) \vee f^*(k)$,

□

Interior operators

one has

$$
\begin{aligned}
f^*(i_Y(k)) &\cong f^*\left(\bigvee\{n \in \mathrm{sub}Y \mid \neg n \vee k \cong 1_Y\}\right) \\
&\cong \bigvee\{f^*(n) \in \mathrm{sub}Y \mid \neg n \vee k \cong 1_Y\} \\
&\leq \bigvee\{f^*(n) \in \mathrm{sub}Y \mid f^*(n) \vee f^*(k) \cong 1_X\} \\
&\leq \bigvee\{p \in \mathrm{sub}X \mid \neg p \vee f^*(k) \cong 1_X\} \\
&\cong i_X(f^*(k)).
\end{aligned}
$$

The category of presheaves

Definition

A presheaf on a small category $\mathbb{C}$ is a functor $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Sets}$.

Thus a presheaf on a category is a contravariant set-valued functor.

Definition

A category with objects all presheaves on an arbitrary small category $\mathbb{C}$ and morphisms all natural transformations between the presheaves is called the category of presheaves and denoted by $\mathrm{Psh}(\mathbb{C})$. $\mathrm{Psh}(\mathbb{C})$ is an example of a functor category.

Let $P \in \mathrm{Psh}(\mathbb{C})$. Then the subobjects of P are the subfunctors of P. That is, $Q \in \mathrm{sub}P$ iff $Q\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Sets}$ is a functor with each QX a subset of PX and each $Qf : QY \to QX$ a restriction of Pf, for all morphisms $f : X \to Y$ of $\mathbb{C}$.

Subpresheaves

Lemma

Let $P \in \mathrm{Psh}(\mathbb{C})$. Then sub$P$ is a complete Heyting algebra with respect to the natural ordering $S \leq T \Leftrightarrow S(X) \subseteq T(X)$ for all $X \in \mathbb{C}$. Indeed,

1. $(S \wedge T)(X) = S(X) \cap T(X)$ for all $X \in \mathbb{C}$;
2. $(S \vee T)(X) = S(X) \cup T(X)$ for all $X \in \mathbb{C}$;
3. $(S \Rightarrow T)(X) = \{x \in P(X) | \left(\forall\ Y \stackrel{f}{\to} X \text{ in } \mathbb{C}\right) (P(f)(x) \in S(Y) \Rightarrow P(f)(x) \in T(Y))\}$ for all $X \in \mathbb{C}$;
4. The bottom subobject of P is the empty functor 0 (with $0(X) = \emptyset\ \forall X \in \mathbb{C}$) and the functor P is the top subobject.

Adjunctions

Lemma

Any morphism $\eta : E \to F$ of presheaves on an arbitrary small category $\mathbb{C}$ induces a functor on the corresponding partially ordered sets of subpresheaves,

$$\eta^* : \mathsf{sub}E \to \mathsf{sub}F$$

by pullback. Moreover, this functor has both a left and a right adjoint:

$$\exists_\eta \dashv \eta^* \dashv \forall_\eta.$$

Interior operator on Psh($\mathbb{C}$)

Proposition

The assignment i that to each subfunctor R of a presheaf E on an arbitrary small category $\mathbb{C}$ associates $i_E(R) = \bigvee\{A \in \mathrm{sub}E \mid \neg A \vee R = E\}$ is a standard interior operator on Psh($\mathbb{C}$).

Proof.

1. The contraction and monotonicity conditions of i are straightforwad.
2. Let $\eta : E \to F$ in Psh($\mathbb{C}$) and $R \in \mathrm{sub}F$. Since $\neg A \vee R = F$ implies

$$E = \eta^*(F) = \eta^*(\neg A \vee R) = \eta^*(\neg A) \vee \eta^*(R) \leq \neg\eta^*(A) \vee \eta^*(R),$$

□

Cont...

one has

$$\begin{aligned}
\eta^*(i_F(R)) &= \eta^*\left(\bigvee\{A \in \mathrm{sub}F \mid \neg A \vee R = F\}\right) \\
&= \bigvee\{\eta^*(A) \in \mathrm{sub}F \mid \neg A \vee R = F\} \\
&\leq \bigvee\{\eta^*(A) \in \mathrm{sub}F \mid \eta^*(\neg A) \vee \eta^*(R) = E\} \\
&\leq \bigvee\{B \in \mathrm{sub}F \mid \neg B \vee \eta^*(R) = E\} \\
&= i_E(\eta^*(R)).
\end{aligned}$$

Sieves

A sieve is a categorical counterpart of a collection of open subsets of a fixed open set in topology.

Definition

Let $\mathbb{C}$ be a small category. A sieve on $Z \in \mathbb{C}$, or a Z-sieve is a subset S of the collection S_Z of all $\mathbb{C}$-morphisms with codomain Z which is closed under right composition, that is, for all $f : Y \to Z$ in S and $g : X \to Y$ in $\mathbb{C}$ then $f \circ g : X \to Z$ in S.

Consequently, a sieve S on Z is a right ideal of morphisms in $\mathbb{C}$, all with codomain Z.
For any object Z there are always at least two Z-sieves $t_Z = \{f : \mathrm{cod}(f) = z\}$ (the maximal sieve) and $\emptyset$ (the empty sieve).

Properties of Sieves

Remark

Sieves on $Z \in \mathbb{C}$ and subfunctors of $\mathrm{Hom}(-, Z)$ are equivalent notions. Indeed, let S be sieve on $Z \in \mathbb{C}$. Then $Q_S \colon \mathbb{C}^{\mathrm{op}} \to \textbf{Sets}$ given by

$$Q_S(Y \xrightarrow{f} X) = Q_S(X) \xrightarrow{Q_S f} Q_S(Y)$$

where $Q_S(X) = \{g \in S \mid g : X \to Z\} \subseteq \mathrm{Hom}(-, Z)$ and $Q_S f(g) = g \circ f$ is a subfunctor of $\mathrm{Hom}(-, Z)$. Conversely, if Q is a subfunctor of $\mathrm{Hom}(-, Z)$, then the set $S_Q = \{f \text{ in } \mathbb{C} \mid (\exists X \in \mathbb{C})(f : X \to Z \in Q(X))\}$ is a sieve on Z.

Definition (Pullback of sieves)

Let $f : X \to Y$ be a morphism in $\mathbb{C}$ and let S be a sieve on Y. We define $f^*(S) = \{g \mid f \circ g \in S\}$ is a sieve on X.

Grothendieck Topology

A Grothendieck topology is a structure on a category $\mathbb{C}$ that makes the objects of $\mathbb{C}$ act like the open sets of a topological space.

Definition

Let $\mathbb{C}$ be a small category. A Grothendieck topology on $\mathbb{C}$ is a map J which assigns to each object X of $\mathbb{C}$ a collection $J(X)$ of sieves on X, in such a way that

- (i) (maximality axiom) the maximal sieve $t_X = \{f \mid \text{cod}(f) = X\}$ is in $J(X)$;
- (ii) (stability axiom) if $S \in J(X)$, then $h^*(S) \in J(Y)$ for any morphism $h : Y \to X$;
- (iii) (transitivity axiom) if $S \in J(X)$ and R is any sieve on X such that $h^*(R) \in J(Y)$ for all $h : Y \to X \in S$, then $R \in J(X)$.

A site is a pair $(\mathbb{C}, J)$ where J is a Grothendieck topology on $\mathbb{C}$. If $S \in J(X)$, one says that S is a covering sieve or that S covers X.

Grothendieck Topology

We say that a sieve S on X covers a morphism $f : Y \to X$ if $f^*(S)$ covers Y ($f^*(S) \in J(Y)$). Consequently, S covers X iff S covers the identity arrow on X. Moreover,

Proposition

A map J which assigns to each object X of $\mathbb{C}$ a collection $J(X)$ of sieves on X is a Grothendieck topology on $\mathbb{C}$ iff the following conditions hold true:

(*i*) if S is a sieve on X and $f \in S$, then S covers f ($f^*(S) \in J(D_f)$);

(*ii*) if S covers a morphism $f : Y \to X$, it also covers the composition $f \circ g$, for any morphism $g : Z \to Y$;

(*iii*) If R is a sieve on X which covers all morphisms of a sieve S on X and S covers a morphism $f : Y \to X$, then R covers f.

Closed sieves

Definition

Let $(\mathbb{C}, J)$ be a site. A sieve S on $X \in \mathbb{C}$ is closed (for J) iff for all morphisms $f : Y \to X$ in $\mathbb{C}$,

$$S \text{ covers } f \Rightarrow f \in S.$$

Note that the terminology " closed" has no intuitive connection with the basic notion of a closed set in point-set topology.

Remark

The property of being closed is stable under pullback, that is, for any sieve S on Y and any morphism $f : X \to Y$,

$$S \text{ is closed } \Rightarrow f^*(S) \text{ is closed }.$$

Indeed, suppose $f^*(S)$ covers $g : W \to X$. Then S covers $f \circ g$. Hence, $f \circ g \in S$ since S is closed. Consequently, $g \in f^*(S)$. Thus $f^*(S)$ is closed.

Sheaves on a Site

Let C be a small Category and J a Grothendieck topology on $\mathbb{C}$. A sheaf P is a functor $P : \mathbb{C}^{\mathrm{op}} \to \mathrm{Sets}$ such that for every covering sieve S on X the inclusion $S \to y(X) = \mathrm{Hom}(-, X)$ induces an isomorphism $\mathrm{Hom}(S, P) \cong \mathrm{Hom}(y(X), P)$.

Remark

For any sheaf E on a site (C, J), the lattice Sub (E) of all subsheaves of E is a complete Heyting algebra. It is also true that any morphism $\phi : E \to F$ of sheaves on an arbitrary small category $\mathbb{C}$ induces a functor on the corresponding partially ordered sets of subpresheaves, $\phi^* : \mathrm{sub}E \to \mathrm{sub}F$ by pullback. Moreover, this functor has both a left and a right adjoint: $\exists_\phi \dashv \phi^* \dashv \forall_\phi$.

Interior operator induced by a Grothendieck topology

Proposition

Let $(\mathbb{C}, J)$ be a fixed site and $\mathrm{Sh}(\mathbb{C}, J)$ the associated category of sheaves. The assignment i^G that to each sieve S on $X \in \mathbb{C}$ associates $i^G(S) = \bigvee\{T \text{ closed sieve on } X \mid T \leq S\}$ is an idempotent interior operator.

Proof.

1. The contraction and monotonicity conditions of i^G are obvious.
2. Let $f : X \to Y$ in $\mathbb{C}$ and $S \in \mathrm{subHom}(-, Y)$. Since the collection of closed sieves is stable under pullback and f^* order preserving and commutes with the join of sieves, one has

□

Cont...

$$
\begin{aligned}
f^*(i_F^G(S)) &= f^*\left(\bigvee\{T \text{ closed sieve on } Y \mid \ T \leq S\}\right)\\
&= \bigvee\{f^*(T) \mid T \text{ closed sieve on } Y \text{ and } T \leq S\}\\
&\leq \bigvee\{f^*(T) \text{ closed sieve on } X \mid \ \text{and } f^*(T) \leq f^*(S)\}\\
&\leq \bigvee\{R \text{ closed sieve on } X \mid \ \text{and } R \leq f^*(S)\}\\
&= i^G(f^*(S)).
\end{aligned}
$$

3. Let T and S be sieves on X such that T is closed and $T \leq S$. Then $T \leq \bigvee\{T \text{ closed sieve on } X \mid T \leq S\} = i^G(S)$. Consequently, $T \in \{R \text{ closed sieve on } X \mid T \leq i^G(S)\}$. Hence, $i^G(S) = \bigvee\{T \text{ closed sieve on } X \mid T \leq S\} \leq \bigvee\{R \text{ closed sieve on } X \mid T \leq i^G(S)\} = i^G(i^G(S))$. Therefore, i^G is idempotent.

Since $i^G(t_X) = t_X$, i^G is also a standard interior operator.

Characterization of morphisms w.r.t i^G

Definition

Let i be an interior operator on $\mathbb{C}$. A $\mathbb{C}$-morphism $f : X \to Y$ is called

(a) i-open if $i_X(f^*(n)) \cong f^*(i_Y(n))$ for all $n \in \text{sub}Y$, that is: the preimage f^* commutes with the interior operator i;

(b) i-closed if $f_*(i_X(m)) \cong i_Y(f_*(m))$ for all $m \in \text{sub}X$, that is: the dual image f_* commutes with the interior operator i;

(c) i-initial if $i_X(m) \cong f^*(i_Y(f_*(m)))$ for all $m \in \text{sub}X$;

(d) i-final if $f_*(i_X(f^*(n))) \cong i_Y(n)$ for all $n \in \text{sub}Y$.

Characterization of morphisms w.r.t i^G

Proposition

Let $f : X \to Y$ be a morphism in $\mathbb{C}$ and i is an idempotent interior operator. Then:

(a) f is i-open iff f maps i-open $\mathcal{M}$-subobjects into i-open $\mathcal{M}$-subobjects;

(b) f is i-closed iff the right adjoint f_* of the preimage f^* maps i-open $\mathcal{M}$-subobjects into i-open $\mathcal{M}$-subobjects;

(c) f is i-initial iff for every i-open subobject m of X, there exists an i-open subobject n of Y such that $m \cong f^*(n)$.

Characterization of morphisms w.r.t i^G

Remark

Since i^G is an idempotent interior operator, one has $f : X \to Y \in \mathbb{C}$ is:

(a) i^G-open iff $\exists_f$ preserves closed sieves;

(b) i^G-closed iff $\forall_f$ preserves closed sieves;

(c) i^G-initial iff for every closed sieve S on X, there exists a closed sieve T on Y such that $S = f^*(T)$.

(d) i^G-final iff for any sieves R and T on Y and a closed sieve P on X such that $f^*(R) \leq P \leq f^*(S)$, one has $R \leq S \leq T$ for some closed sieve S on X.

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Thank you for your attention!