Solitary wave solutions and classical symmetry reductions of a generalized bi-dimensional nonlinear wave equation in engineering physics

Oke Davies Adeyemo, Chaudry Masood Khalique

International Institute for Symmetry Analysis and Mathematical Modelling
Department of Mathematical Sciences
North-West University, Mafikeng Campus
REPUBLIC OF SOUTH AFRICA

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Preamble

We study the generalized bi-dimensional nonlinear wave equation

$$\beta u_{xxxx} + \alpha u_{xt} + 2\gamma u_x^2 + 2\gamma u u_{xx} - \gamma u_{yy} = 0$$

(1) We compute the *Lie point symmetries*

(3) We then obtain *symmetry reduction and exact solutions*

(4) We derive *Conservation laws*

(5) Concluding remarks.

Introduction

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Nonlinear partial differential equations (NLPDEs) model physical phenomena in the fields of mathematics, physics and engineering.

Such phenomena occur in oceanography, aerospace industry, meteorology, nonlinear mechanics, biology, population ecology, plasma physics, fluid mechanics to mention but a few.

In order to really understand these physical phenomena it is of immense importance to solve these NLPDEs which govern these aforementioned phenomena.

Therefore, in the subsequent part of this talk, we consider one of these equations.

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Literature review

The Yu-Toda-Sasa-Fukuyama system expresssed as

Y. Hu, H. Chen, Z. Dai, New kink multi-soliton solutions for the (3+1)-dimensional potential-Yu-Toda-Sasa-Fukuyama equation, Appl. Math. Comput., 234 (2014) 548–556,

$$(-4v_t + \Phi(v)v_z)_x + 3v_{yy} = 0,$$
$$\Phi(v) = \partial_x^2 + 4v + 2v_x\partial_x^{-1},$$

transforms to

$$u_{xxxz} - 4u_{xt} + 4u_x u_{xz} + 2u_{xx}u_z + 3u_{yy} = 0, \quad (1)$$

for $v : \mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z \times \mathbb{R}_t \to \mathbb{R}$ such that potential $v = u_x$.

Equation (1) is referred to as the (3+1)-dimensional Yu-Toda-Sasa-Fukuyama equation which in turn is regarded as an adjunct of the Bogoyavlenskii-Schiff equation. Predominantly, Hu et al. in the previously given reference, exploited transformation of unspecified functions to alter (1) into an incorporated equation of distinctively two bilinear structures.

Besides, the authors by utilizing Darvishi's scheme procured some modish kink multi-soliton solutions due to the engagement of homoclinic test technique and three-wave approach respectively. It is noteworthy to assert here that equation (1) is never an integrable system

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Moreover, the non-travelling wave solution was achieved by way of auto-Bäcklund transformation as well as the generalized projective Riccati equation techniques

Z. Yan, New families of nontravelling wave solutions to a new (3+1)-dimensional potential-YTSF equation, Phys. Lett. A, 318 (2003) 78–83,

X. Zeng, Z. Dai, D. Li, New periodic soliton solutions for the (3+1)-dimensional potential-YTSF equation, Chaos Soliton Fract., 42 (2009) 657–661,

J. Schff, Painlevé Transendent, Their Asymptotics and Physical Applications, Plenum, New York, NY, USA, 1992.

Furthermore, some periodic solutions along with soliton-like solutions for equation (1) were gained via Hirota bilinear, tanh-coth, exp-function, homoclinic test, as well as the extended homoclinic test techniques. See

A. Wazwaz, Multiple-soliton solutions for the Calogero-Bogoyavlenskii-Schiff, Jimbo-Miwa and YTSF equations, Appl. Math. Comput., 203 (2008) 592–597,

A. Borhanifar, M.M. Kabir, New periodic and soliton solutions by application of exp-function method for nonlinear evolution equations, J. Comput. Appl. Math., 229 (2009) 158–167,

Z. Guo, J. Yu, Multiplicity results on period solutions to higher dimensional differential equations with multiple delays, J. Dyn. Differ. Equ., 23 (2011) 1029–1052

and the ealier given references.

Lately, some closed-form solutions to the (3+1)-dimensional potential Yu-Toda-Sasa-Fukuyama (1) were secured by Darvishi and Najafi

M.T. Darvishi, M. Najafi, A modification of EHTA to solve the (3+1)-dimensional potential-YTFS equation, Chin. Phys. Lett., 28 (2011) 040202

through the engagement of the modified extended homoclinic test technique. In the same vein, some solutions to (1) were constructed via the exploitation of the modified simple equation technique

E.M.E. Zayed, A.H. Arnous, Exact solutions of the nonlinear ZK-MEW and the potential YTSF equations using the modified simple equation method. AIP Conference Proceedings, 1479, American Institute of Physics, 2012.

In addition, some analytic travelling wave solutions to (1) were obtained by invoking the $(G'/G, 1/G)-$expansion, tan-hyperbolic and symmetry methods

E.M.E. Zayed, S.A. Hoda Ibrahim, The two variable $(G'/G, 1/G)$- expansion method for finding exact traveling wave solutions of the (3+1)-dimensional nonlinear potential Yu-Toda-Sasa-Fukuyama equation: International Conference on Advanced Computer Science and Electronics Information (ICACSEI 2013), Atlantis Press, (2013) 388–392,

S. Sahoo, S.S. Ray, Lie symmetry analysis and exact solutions of (3+1)-dimensional Yu-Toda-Sasa-Fukuyama equation in mathematical physics, Comput. Math. Appl., 73 (2017) 253–260.

In

S.J. Chen, Y.H. Yin, W.X. Ma, X, Lü, Abundant exact solutions and interaction phenomena of the (2+1)-dimensional YTSF equation, Anal. Math. Phys., 9 (2019) 2329–2344, Chen et al. introduced (1) in their work as

$$w_{xxxz} - 4w_{xt} + 4w_x w_{xz} + 2w_{xx} w_z + 3w_{yy} = 0. \tag{2}$$

The authors employed the transformation $x - z \to x$, $y \to y$ along with $t \to t$, and so equation (2) is transformed into

$$-4w_{xt} - w_{xxxx} - 6w_x w_{xx} + 3w_{yy} = 0. \tag{3}$$

Instigating $u(t, x, y) = \frac{\partial w(t,x,y)}{\partial x}$ as a potential function, the induction of (3) with respect to x recasts it to

$$-4u_{xt}-u_{xxxx}-6u_x^2-6uu_{xx}+3u_{yy}=0, \quad (4)$$

which is called the (2+1)-dimensional YTSFe. They further enunciated the interconnectivity between some notable equations, viz., KP and BS equations can be obtained from KdV equation. Also they can equally be expanded to the (3+1)-dimensional YTSFe and conclusively, equation (1) can be transmuted to (4).

Furthermore, in their investigation the authors examined copious closed-form solutions of (4) via the infusement of symbolic computation hinged on Hirota bilinear formulation to achieve lump solutions as well as interraction solutions of (4).

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Our work here investigates a generalized version of (4) which we call the generalized bi-dimensional nonlinear wave equation (2D-gNLWE)

$$Q \equiv \beta u_{xxxx} + \alpha u_{xt} + 2\gamma u_x^2 + 2\gamma u u_{xx} - \gamma u_{yy} = 0, \tag{5}$$

with α, γ and β regarded as nonzero real constants where $\alpha = 4$, $\gamma = 3$ and $\beta = 1$ relative to (4).

Lie group analysis

Lie point symmetries of 2D-gNLWE

The commensurating Lie algebra of infinitesimal generators of the 2D-gNLWE (5) is spanned by the vector field G recounted as

$$G = \xi^1(t,x,y,u)\frac{\partial}{\partial x} + \xi^2(t,x,y,u)\frac{\partial}{\partial y} + \xi^3(t,x,y,u)\frac{\partial}{\partial t} + \phi(t,x,y,u)\frac{\partial}{\partial u}$$

in which the coefficient functions $\xi^1(t,x,y,u), \xi^2(t,x,y,u), \xi^3(t,x,y,u)$ alongside $\phi(t,x,y,u)$ are to be decided.

The attributed formula for the prolongation of a fourth-order 2D-gNLWE (5) prescribed as $Pr^{(4)}G$ is conveyed as

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The attributed formula for the prolongation of a fourth-order 2D-gNLWE (5) prescribed as $Pr^{(4)}G$ is conveyed as

$$Pr^{(4)}G = G + \phi^x \frac{\partial}{\partial u_x} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{yy} \frac{\partial}{\partial u_{yy}} + \phi^{xxxx} \frac{\partial}{\partial u_{xxxx}}. \quad (6)$$

Envisaging invariance condition to (5), we obtain

$$Pr^{(4)}G \left(\beta u_{xxxx} + \alpha u_{xt} + 2\gamma u_x^2 + 2\gamma u u_{xx} - \gamma u_{yy}\right)\Big|_{Q=0} = 0, \quad (7)$$

which gives

$$\left\{\alpha\phi^{xt} - 2\gamma((\phi^x)^2 - u\phi^{xx} - \phi u_{xx}) - \gamma\phi^{yy} + \beta\phi^{xxxx}\right\} |_{Q=0} = 0, \quad (8)$$

where ϕ^x, ϕ^{xx}, ϕ^{xt}, ϕ^{yy}, and ϕ^{xxxx} are regarded as coefficient functions of $Pr^{(4)}G$ in equation (6).

The coefficient functions are explicated as

$$\begin{aligned}
\phi^{x} &= D_x(\phi) - u_t D_x(\xi^1) - u_x D_x(\xi^2) - u_y D_x(\xi^3),\\
\phi^{y} &= D_y(\phi) - u_t D_y(\xi^1) - u_x D_y(\xi^2) - u_y D_y(\xi^3),\\
\phi^{t} &= D_t(\phi) - u_t D_t(\xi^1) - u_x D_t(\xi^2) - u_y D_t(\xi^3),\\
\phi^{xx} &= D_x(\phi^{x}) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2) - u_{xy} D_x(\xi^3),\\
\phi^{yy} &= D_y(\phi^{y}) - u_{ty} D_y(\xi^1) - u_{xy} D_y(\xi^2) - u_{yy} D_y(\xi^3),\\
\phi^{xt} &= D_t(\phi^{x}) - u_{tx} D_t(\xi^1) - u_{xx} D_t(\xi^2) - u_{xy} D_t(\xi^3),\\
\phi^{xxxx} &= D_x(\phi^{xxx}) - u_{txxx} D_x(\xi^1) - u_{xxxx} D_x(\xi^2) - u_{xxxy} D_x(\xi^3), \qquad (9)
\end{aligned}$$

where D_t, D_x, D_y are the total derivatives with respect to t, x and y, accordingly.

The total derivatives involved are designated as

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} + \cdots ,$$
$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + \cdots ,$$
$$D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{ty} \frac{\partial}{\partial u_t} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y} + \cdots . \qquad (10)$$

We expand equation (8) with the help of relations in (9) and (10), and thereafter equate the various coefficients of the partial derivatives of u to zero.

Consequently, the following twenty determining equations of 2D-gNLWE (5) are obtained:

$$\xi^1_u = 0, \quad \xi^2_x = 0,\ \xi^3_x = 0,\ \xi^2_u = 0,\ \xi^3_u = 0,\ \xi^3_y = 0,$$
$$\phi_{xx} = 0,\ \phi_{xy} = 0,\ \phi_{ux} = 0,\ \phi_{uy} = 0,\ \phi_{uu} = 0,$$
$$2\xi^1_x + \phi_u = 0,\ \xi^2_y + \phi_u = 0,\ \alpha\phi_{xt} - \gamma\phi_{yy} = 0,$$
$$\phi_{yyy} = 0,\ \alpha\phi_{ut} + 4\gamma\phi_x = 0,\ \xi^1_{yy} - 2\phi_x = 0,$$
$$2\xi^3_t + 3\phi_u = 0,\ \alpha\xi^2_t - 2\gamma\xi^1_y = 0,$$
$$\alpha\xi^1_t + 2\gamma(u\phi_u - \phi_u) = 0.$$

The solution of the obtained system yields the following

$$\xi^1 = \frac{2\gamma}{\alpha} x F_3'(t) - \frac{1}{2} C_1 x + y^2 F_3''(t) + \frac{2\gamma}{\alpha} y F_2(t) + \frac{\alpha}{2\gamma} y C_2 + \frac{2\gamma}{\alpha} F_1(t) + C_4,$$
$$\xi^2 = \frac{4\gamma}{\alpha} y F_3'(t) - C_1 y + \frac{4\gamma^2}{\alpha^2} \int\int F_2'(t) dt dt + C_2 t + C_3,$$
$$\xi^3 = \frac{6\gamma}{\alpha} F_3(t) - \frac{3}{2} C_1 t + C_5,$$
$$\phi = x F_3''(t) + \left(-\frac{4\gamma}{\alpha} F_3'(t) + C_1\right) u + \frac{\alpha}{2\gamma} y^2 F_3'''(t) + y F_2'(t) + F_1'(t)$$

with F_1, F_2 and F_3 arbitrary functions depending on t.

Hence, the solution occasioned an eight-dimensional Lie algebra L^8 admitted by 2D-gNLWE (5) and are spanned by the succeeding vector fields, namely:

$$
\begin{aligned}
G_1 &= \frac{\partial}{\partial t}, \\
G_2 &= \frac{\partial}{\partial x}, \\
G_3 &= \frac{\partial}{\partial y}, \\
G_4 &= \alpha y \frac{\partial}{\partial x} + 2\gamma t \frac{\partial}{\partial y}, \\
G_5 &= 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} - 2u \frac{\partial}{\partial u},
\end{aligned}
$$

$$
\begin{aligned}
G_6 &= 2\gamma F_1(t)\frac{\partial}{\partial x} + \alpha F_1'(t)\frac{\partial}{\partial u},\\
G_7 &= 2\gamma\alpha y F_2(t)\frac{\partial}{\partial x} + 4\gamma^2\tilde{F}_2(t)\frac{\partial}{\partial y} + \alpha^2 y F_2'(t)\frac{\partial}{\partial u},\\
G_8 &= 12\gamma^2 F_3(t)\frac{\partial}{\partial t} + 2\gamma(\alpha y^2 F_3''(t) + 2\gamma x F_3'(t))\frac{\partial}{\partial x} + 8\gamma^2 y F_3'(t)\frac{\partial}{\partial y}\\
&\quad + (\alpha^2 y^2 F_3'''(t) + 2\alpha\gamma x F_3''(t) - 8\gamma^2 u F_3'(t))\frac{\partial}{\partial u},
\end{aligned}
\tag{11}
$$

where $\tilde{F}_2(t) = \int F_2(t)dt$.

Symmetry reductions and exact solutions

Next, we put the symmetries to use in the reduction process for 2D-gNLWE (5). We start by considering the symmetry generator $G = G_1 + G_2 + \nu G_3$ with nonzero constant ν and convert the 2D-gNLWE (5) to a NLPDE whose dependency is on two independent variables. In consequence, solving the related Lagrangian system of G, one thereby procures the invariants

$$f = t - \nu y, \ g = x - y, \ \theta = u. \tag{12}$$

Endorsing θ in (12) as the contemporary dependent variable as well as f, g as independent variables, 2D-gNLWE (5) then alters to the NLPDE obtained as

$$\gamma\nu^2\theta_{ff} + 2\gamma\nu\theta_{fg} - \alpha\theta_{fg} - 2\gamma\theta_g^2 - 2\gamma\theta\theta_{gg} + \gamma\theta_{gg} - \beta\theta_{gggg} = 0. \quad (13)$$

Next we entrench the Lie point symmetries of (13) and engage them

to alter it to an ordinary differential equation (ODEQ). Equation (13) has the following symmetries:

$$\begin{aligned}\Upsilon_1 &= \frac{\partial}{\partial g},\\ \Upsilon_2 &= \frac{\partial}{\partial f},\\ \Upsilon_3 &= 2\gamma(2\gamma\nu^2 g + 2\gamma\nu f - \alpha f)\frac{\partial}{\partial g} + 8\gamma^2\nu^2 f\frac{\partial}{\partial f}\\ &\quad - (8\gamma^2\nu^2\theta - 4\gamma\alpha\nu + \alpha^2)\frac{\partial}{\partial\theta}\end{aligned}$$

and the linear combination $\Upsilon = \Upsilon_1 + \xi \Upsilon_2$, with ξ, a nonzero constant yields the two invariants: $\zeta = f - \xi g, \ \Phi = \theta$, which produces a group invariant solution Φ, with the dependency of Φ on ζ.

Utilization of the computed invariants converts 2D-gNLWE (5) to the fourth-order nonlinear ordinary differential equation (NODE)

$$(\alpha\xi+\gamma\nu^2-2\gamma\nu\xi+\gamma\xi^2)\Phi''(\zeta)-2\gamma\xi^2\Phi'^2(\zeta)-2\gamma\xi^2\Phi(\zeta)\Phi''(\zeta)-\beta\xi^4\Phi''''(\zeta)=0. \tag{14}$$

Next, we secure periodic and non-topological 1-soliton solutions of 2D-gNLWE (5) by direct integration of the found equation.

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Next, we secure periodic and non-topological 1-soliton solutions of 2D-gNLWE (5) by direct integration of the found equation.

Carrying out integration of (14) once with respect to ζ we achieve

$$(\alpha\xi+\gamma\nu^2-2\gamma\nu\xi+\gamma\xi^2)\Phi'(\zeta)-2\gamma\xi^2(\Phi(\zeta)\Phi'(\zeta))-\beta\xi^4\Phi'''(\zeta)+C_0=0, \quad (15)$$

with C_0, an integration constant.

To compute Jacobi elliptic cosine solutions of 2D-gNLWE (5), we assume $C_0=0$ and integrate (15) with respect to ζ to get

$$(\alpha\xi+\gamma\nu^2-2\gamma\nu\xi+\gamma\xi^2)\Phi(\zeta)-\gamma\xi^2\Phi^2(\zeta)-\beta\xi^4\Phi''(\zeta)+C_1=0, \quad (16)$$

where C_1 is taken as a constant of integration.

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where C_1 is taken as a constant of integration.

Multiplying equation (16) by $\Phi'(\zeta)$ and integrating the resulting equation with respect to ζ, we gain

$$\frac{1}{2}(\alpha\xi + \gamma\nu^2 - 2\gamma\nu\xi + \gamma\xi^2)\Phi^2(\zeta) - \frac{1}{3}\gamma\xi^2\Phi^3(\zeta) - \frac{1}{2}\beta\xi^4\Phi'^2(\zeta) + C_1\Phi(\zeta) + C_2 = 0. \tag{17}$$

In this case we observe equation (17) with $C_1 = C_2 \neq 0$, thus we achieve

$$\Phi'^2(\zeta) + \frac{2\gamma}{3\beta\xi^2}\Phi^3(\zeta) + (2\gamma\nu\xi - \alpha\xi - \gamma\nu^2 - \gamma\xi^2)\Phi^2(\zeta) - \frac{2C_1}{\beta\xi^4}\Phi(\zeta) - \frac{2C_2}{\beta\xi^4} = 0. \tag{18}$$

We note here that the solutions of equation (18) can be conveyed via Jacobi elliptic function

N.A. Kudryashov, Analytical theory of nonlinear differential equations. Moskow-Igevsk: Institute of Computer Investigations; 2004.

Thus, suppose that the cubic equation

$$\Phi^3(\zeta)+\frac{3\beta\xi^2(2\gamma\nu\xi-\alpha\xi-\gamma\nu^2-\gamma\xi^2)}{2\gamma}\Phi^2(\zeta)-\frac{3C_1}{\gamma\xi^2}\Phi(\zeta)-\frac{3C_2}{\gamma\xi^2}=0 \quad (19)$$

has s_1, s_2 and s_3 as its roots such that $s_1 > s_2 > s_3$, then equation (18) can be written as

$$\Phi'^2(\zeta)+\frac{2\gamma}{3\beta\xi^2}(\Phi-s_1)(\Phi-s_2)(\Phi-s_3)=0. \quad (20)$$

Thus, the general solution to equation (14) in terms of Jacobi elliptic cosine amplitude function is

$$\Phi(\zeta) = s_2 + (s_1 - s_2)\mathrm{cn}^2\left(\sqrt{\frac{\gamma}{6\beta\xi^2}(s_1 - s_3)}\,\zeta; S^2\right),\ S^2 = \frac{s_1 - s_2}{s_1 - s_3}, \quad (21)$$

where $0 \le S^2 \le 1$. The function (cn) in equation (21) denotes the cosine amplitude elliptic function. Now reverting to the basic variables, one secures the solution of the 2D-gNLWE (5) as

$$u(t,x,y) = s_2 + (s_1 - s_2)\mathrm{cn}^2\left(\sqrt{\frac{\gamma}{6\beta\xi^2}(s_1 - s_3)}\,\zeta; S^2\right),\ S^2 = \frac{s_1 - s_2}{s_1 - s_3} \quad (22)$$

with $\zeta = t - \xi x + (\xi - \nu)y$.

It is well known that taking special limits of the Jacobi elliptic function degenerates it into some other mathematical functions including hyperbolic and trigonometric functions. We engage that notion in this regard to gain more solutions of (5) from the established Jacobi elliptic solution (22).

In consequence, having known that $0 \leq S^2 \leq 1$, we first consider the case of S^2 approaching 1 ($S^2 \to 1$) and so, we secure a bright soliton solution

$$u(t,x,y) = s_2 + (s_1 - s_2)\,\mathrm{sech}^2\left(\sqrt{\frac{\gamma}{6\beta\xi^2}(s_1 - s_3)}\,\zeta\right). \tag{23}$$

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where $\zeta = t - \xi x + (\xi - \nu)y$.

Further, we examine the elliptic solution as S^2 approaches 0. In doing so, we achieve a trigonometric solution of (5) as

$$u(t,x,y) = s_2 + (s_1 - s_2)\cos^2\left(\sqrt{\frac{\gamma}{6\beta\xi^2}(s_1 - s_3)}\,\zeta\right), \tag{24}$$

with $\zeta = t - \xi x + (\xi - \nu)y$. Thus, we establish for equation (5) elementary function solutions from Jacobi elliptic solution (22) via special limits of the elliptic function.

Next, we present group-invariant solutions of 2D-gNLWE (5) using the listed generators in (11).

Further, we examine the elliptic solution as S^2 approaches 0. In doing so, we achieve a trigonometric solution of (5) as

$$u(t,x,y) = s_2 + (s_1 - s_2)\cos^2\left(\sqrt{\frac{\gamma}{6\beta\xi^2}(s_1 - s_3)}\,\zeta\right), \tag{24}$$

with $\zeta = t - \xi x + (\xi - \nu)y$. Thus, we establish for equation (5) elementary function solutions from Jacobi elliptic solution (22) via special limits of the elliptic function.

Next, we present group-invariant solutions of 2D-gNLWE (5) using the listed generators in (11).

Group-invariant solution via generator G_1

Lie symmetry $G_1 = \partial/\partial t$ gives the Lagrangian system structured as

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dy}{0} = \frac{du}{0}, \tag{25}$$

whose invariants are calculated as $X = x$ and $Y = y$ with group-invariant $u = Q(X, Y)$.

Substituting the function assigned to u into (5) gives its reduced form as

$$\gamma Q_{YY} - 2\gamma Q_X^2 - 2\gamma Q Q_{XX} - \beta Q_{XXXX} = 0. \tag{26}$$

Hence, the solution of (26) yields a dark soliton solution presented as

$$Q(X, Y) = \frac{1}{2\gamma A_1^2}\left(\gamma A_2^2 + 8\beta A_1^4\right) - \frac{6\beta A_1^2}{\gamma}\tanh\left(A_1 X + A_2 Y + A_0\right)^2, \tag{27}$$

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where A_0, A_1 and A_2 are arbitrary constants. On retrograding to the fundamental variables, we secure a hyperbolic solution of (5) as

$$u(t,x,y) = \frac{1}{2\gamma A_1^2}\left(\gamma A_2^2 + 8\beta A_1^4\right) - \frac{6\beta A_1^2}{\gamma}\tanh\left(A_1 x + A_2 y + A_0\right)^2. \quad (28)$$

Examining equation (26) using Lie theoretic approach furnishes symmetries

$$S_1 = \frac{\partial}{\partial X}, \quad S_2 = \frac{\partial}{\partial Y}, \quad S_3 = \frac{1}{2}X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} - Q\frac{\partial}{\partial Q}. \quad (29)$$

Engaging S_1, we obtain the solution $Q(X,Y) = \Phi(\zeta)$, with $\zeta = Y$, which transforms (5) to ODEQ $\Phi''(\zeta) = 0$.

where A_0, A_1 and A_2 are arbitrary constants. On retrograding to the fundamental variables, we secure a hyperbolic solution of (5) as

$$u(t,x,y) = \frac{1}{2\gamma A_1^2}\left(\gamma A_2^2 + 8\beta A_1^4\right) - \frac{6\beta A_1^2}{\gamma}\tanh\left(A_1 x + A_2 y + A_0\right)^2. \quad (28)$$

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Solving the equation gives

$$u(t, x, y) = A_0 y + A_1, \tag{30}$$

with integration constants A_0 and A_1. In the same vein, for S_2, we have the solution calculated as $Q(X, Y) = \Phi(\zeta)$, with $\zeta = X$.

Using the function reduces (5) to NODE

$$2\gamma\Phi'(\zeta)^2 + 2\gamma\Phi(\zeta)\Phi''(\zeta) + \beta\Phi''''(\zeta) = 0. \tag{31}$$

Integration of equation (31) twice, yields second order ODEQ obtained as

$$A_1\zeta + A_2 + \gamma\Phi(\zeta)^2 + \beta\Phi''(\zeta) = 0. \tag{32}$$

Solving the equation gives

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Letting $A_1 = A_2 = 0$, integrating the product of the resulting equation and $\Phi'(\zeta)$ and then retrogading to the basic variables yields

$$u(t,x,y) = -\sqrt[3]{\frac{2}{\gamma}}\,\wp\left(6^{-1/2}\sqrt{\frac{2^{2/3}\gamma^{4/3}}{2^{1/3}\gamma^{2/3}\beta}}x + C_0, g_1, g_2\right), \qquad (33)$$

where $\wp$ is a Weierstrass elliptic function solution

N.I. Akhiezer, Elements of The Theory of Elliptic Functions, American Mathematical Soc., Providence, Rhode Island, USA, 1990

of 2D-gNLWE (5) with elliptic invariants $g_1 = g_2 = 0$ and integration constant C_0.

Besides, on letting $A_1 = 0, A_2 \neq 0$, adopting the same steps earlier given and solving the resultant equation, we have

$$u(t,x,y) = -\sqrt[3]{\frac{2}{\gamma}}\,\wp\left(6^{-1/2}\sqrt{\frac{2^{2/3}\gamma^{4/3}}{2^{1/3}\gamma^{2/3}\beta}}x + C_0, g_1, g_2\right), \tag{34}$$

where $g_1 = -6\sqrt[3]{2/\gamma}A_2$ and $g_2 = 0$. We notice that elliptic functions (33) as well as (34) are both steady-state Weierstrass elliptic solution of (5).

Next, we contemplate the linear combination of S_1 and S_2 as $S = c_0\partial/\partial X + c_1\partial/\partial Y$, solution of related characteristic equation is

$$\zeta = c_0Y - c_1X, \text{ where function } Q(X,Y) = \Phi(\zeta). \tag{35}$$

Besides, on letting $A_1 = 0, A_2 \neq 0$, adopting the same steps earlier given and solving the resultant equation, we have

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$$\zeta = c_0 Y - c_1 X, \text{ where function } Q(X,Y) = \Phi(\zeta). \qquad (35)$$

Inserting the variables acquired in (35) into (5), one gets NODE

$$\gamma c_0^2\Phi''(r) - 2\gamma c_1^2\Phi(r)\Phi''(r) - 2\gamma c_1^2\Phi'(r)^2 - \beta c_1^4\Phi''''(r) = 0. \qquad (36)$$

On integrating (36) thrice, with the constants of integration assumed to be zero gives a first-order NODE. Thus, solution to the secured differential equation then furnishes

$$u(t,x,y) = \frac{3}{2c_1^2}\left\{c_0^2 - c_0^2\tanh^2\left[\frac{1}{2}\left(\sqrt{3\gamma}\,c_0A_0 - \frac{\sqrt{\gamma}c_0\,(c_0y - c_1x)}{\sqrt{\beta}c_1^2}\right)\right]\right\} \qquad (37)$$

with constant of integration A_0. Now, we consider the third generator S_3.

Inserting the variables acquired in (35) into (5), one gets NODE

$$\gamma c_0^2\Phi''(r) - 2\gamma c_1^2\Phi(r)\Phi''(r) - 2\gamma c_1^2\Phi'(r)^2 - \beta c_1^4\Phi''''(r) = 0. \quad (36)$$

On integrating (36) thrice, with the constants of integration assumed to be zero gives a first-order NODE. Thus, solution to the secured differential equation then furnishes

$$u(t,x,y) = \frac{3}{2c_1^2}\left\{c_0^2 - c_0^2\tanh^2\left[\frac{1}{2}\left(\sqrt{3\gamma}\,c_0A_0 - \frac{\sqrt{\gamma}c_0\,(c_0y - c_1x)}{\sqrt{\beta}c_1^2}\right)\right]\right\} \quad (37)$$

with constant of integration A_0. Now, we consider the third generator S_3.

Thus, solution of S_3 gives related invariant along with group-invariant accordingly as

$$\zeta = \frac{Y}{X^2}, \text{ and } Q(X,Y) = X^{-2}\Phi(\zeta). \tag{38}$$

Engaging the function and variables in (38) reduces (5) to NODE

$$\begin{aligned}&\gamma\Phi''(\zeta) - 20\gamma\Phi'(\zeta)^2 - 44\gamma\zeta\Phi(\zeta)\Phi'(\zeta) - 120\beta\Phi(\zeta) - 8\gamma\zeta^2\Phi'(\zeta)^2 \\ &- 720\beta\zeta\Phi'(\zeta) - 8\gamma\zeta^2\Phi(\zeta)\Phi''(\zeta) - 732\beta\zeta^2\Phi''(\zeta) - 208\beta\zeta^3\Phi'''(\zeta) \\ &- 16\beta\zeta^4\Phi''''(\zeta) = 0.\end{aligned} \tag{39}$$

Group-invariant solution via generator G_2

Generator $G_2 = \partial/\partial x$ secures the solution $u = Q(T, Y)$, where $T = t$ and $Y = y$. In consequence, we transform (5) to linear partial differential equation (LPDEQ) $Q_{YY}(T, Y) = 0$. Therefore, solving the LPDEQ gives

$$Q(T, Y) = f_1(T)Y + f_2(T). \tag{40}$$

Back-substitution to the fundamental variables gives the solution of (5) as

$$u(t, x, y) = f_1(t)y + f_2(t), \tag{41}$$

where arbitrary functions $f_1(t)$ as well as $f_2(t)$ are both dependent on t.

Group-invariant solution via generator G_3

We transform equation (5) to a NLPDE through $G_3 = \partial/\partial y$ that reads

$$\alpha Q_{TX} + 2\gamma Q_X^2 + 2\gamma QQ_{XX} + \beta Q_{XXXX} = 0, \tag{42}$$

which is accomplished through the use of group-invariant $u = Q(T, X)$ where $T = t$ and $X = x$.

Hence, we secure a solution of (5) here as a complex bright soliton

$$u(t,x,y) = \frac{1}{2\gamma A_2}\left(8A_2^3\beta - A_1\alpha\right) + \frac{6A_2^2\beta\cosh\left(\frac{1}{2}i\pi - A_2x - A_1t - A_0\right)^2}{\gamma\cosh\left(A_2x + A_1t + A_0\right)^2}, \tag{43}$$

Group-invariant solution via generator G_3

We transform equation (5) to a NLPDE through $G_3 = \partial/\partial y$ that reads

$$\alpha Q_{TX} + 2\gamma Q_X^2 + 2\gamma Q Q_{XX} + \beta Q_{XXXX} = 0, \tag{42}$$

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$$u(t,x,y) = \frac{1}{2\gamma A_2}\left(8A_2^3\beta - A_1\alpha\right) + \frac{6A_2^2\beta\cosh\left(\frac{1}{2}\mathrm{i}\pi - A_2x - A_1t - A_0\right)^2}{\gamma\cosh\left(A_2x + A_1t + A_0\right)^2}, \tag{43}$$

where A_0, A_1 and A_2 are arbitrary constant. Further investigation of (43) reveals that it admits two symmetries obtained as $S_1 = \partial/\partial X$ and $S_2 = \partial/\partial T$. On invoking S_1, one obviously gets a trivial solution. We then use S_2 which produces $Q(X, Y) = \Phi(\zeta)$, with $\zeta = X$ and so reduces (5) to NODE obtained as

$$4\gamma\Phi(\zeta)\Phi'(\zeta) + 2\gamma\Phi(\zeta)\Phi''(\zeta) + \beta\Phi''''(\zeta) = 0. \tag{44}$$

Now, we contemplate the linear combination of the two generators as $S = c_1S_1 + c_2S_2$. Similarity solution of S then gives $Q(T, X) = \Phi(\zeta)$ with $\zeta = c_2X - c_1T$. On utilizing the invariants, we further reduce (5) to

$$2\gamma c_0^2\Phi(\zeta)\Phi''(\zeta) + 4\gamma c_0\Phi(\zeta)\Phi'(\zeta) - \alpha c_0c_1\Phi''(\zeta) + \beta c_0^4\Phi''''(\zeta) = 0. \tag{45}$$

where A_0, A_1 and A_2 are arbitrary constant. Further investigation of (43) reveals that it admits two symmetries obtained as $S_1 = \partial/\partial X$ and $S_2 = \partial/\partial T$. On invoking S_1, one obviously gets a trivial solution. We then use S_2 which produces $Q(X, Y) = \Phi(\zeta)$, with $\zeta = X$ and so reduces (5) to NODE obtained as

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$$2\gamma c_0^2\Phi(\zeta)\Phi''(\zeta) + 4\gamma c_0\Phi(\zeta)\Phi'(\zeta) - \alpha c_0c_1\Phi''(\zeta) + \beta c_0^4\Phi''''(\zeta) = 0. \tag{45}$$

Lie symmetry generator G_5 furnishes the group-invariant alongside its invariants as

$$u = t^{-2/3}Q(X, Y), \text{ where } X = \frac{x}{t^{1/3}} \text{ and } Y = \frac{y}{t^{2/3}}. \quad (46)$$

Using the group-invariant in (5) transforms it to the PDE

$$3\alpha Q_X - 6\gamma Q_X^2 - 6\gamma QQ_{XX} + 2\alpha YQ_{XY} + \alpha XQ_{XX} + 3\gamma Q_{YY} - 3\beta Q_{XXXX} = 0. \quad (47)$$

Performing symmetry analysis on equation (47) yields three generators

$$S_1 = \frac{\partial}{\partial X} + \frac{\alpha}{6\gamma}\frac{\partial}{\partial Q}, \quad S_2 = \frac{\alpha}{3\gamma}Y\frac{\partial}{\partial X} + \frac{\partial}{\partial Y} - \frac{\alpha^2}{18\gamma^2}Y\frac{\partial}{\partial Q},$$
$$S_3 = \left(\frac{1}{2}X + \frac{\alpha}{4\gamma}Y^2\right)\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + \frac{1}{8\gamma^2}\left(2\alpha\gamma X - \alpha^2 Y^2 - 8\gamma^2 Q\right)\frac{\partial}{\partial Q}. \tag{48}$$

Next, on involving S_1 and taking the usual steps, we obtain $Q(X, Y) = \Phi(\zeta) + \alpha/6\gamma X$ with $\zeta = Y$. Using the obtained function, we reduce (5) to the ODEQ presented as

$$9\gamma^2\Phi''(\zeta) + \alpha^2 = 0. \tag{49}$$

Solving equation (49) and reverting to basic variables gives the solution

$$u(t,x,y) = t^{-2/3}\left\{\frac{\alpha x}{6\gamma t^{1/3}} - \frac{\alpha^2 y^2}{18\gamma^2 t^{4/3}} + A_0\frac{y}{t^{2/3}} + A_1\right\}, \qquad (50)$$

where A_0 and A_1 are integration constants. We explore generator S_2 and solving the corresponding Lagrangian system we obtain

$$Q(X,Y) = \Phi(\zeta) - \frac{\alpha}{6\gamma}X, \text{ where } \zeta = \alpha Y^2 - 6\gamma X. \qquad (51)$$

On utilizing (51), we further reduce equation (5) to NODE expressed as

$$72\alpha\gamma^2\Phi'(\zeta) + 36\alpha\gamma^2\Phi''(\zeta) + 648\gamma^4\Phi(\zeta)\Phi''(\zeta) + 648\gamma^4\Phi'(\zeta)^2 + 11664\beta\gamma^5\Phi''''(\zeta) + 2\alpha^2 = 0. \qquad (52)$$

We now contemplate generator S_3 and so we have the similarity solution

$$Q(X,Y) = Y^{-1}\Phi(\zeta) + \frac{\alpha}{6\gamma}X - \frac{\alpha^2}{18\gamma^2}Y^2, \text{ where } \zeta = \frac{1}{6\gamma\sqrt{Y}}\left(6\gamma X - \alpha Y^2\right). \tag{53}$$

Engaging the function achieved from (53) in (5), then one obtains

$$\gamma\zeta\Phi''(\zeta) - 48\gamma\Phi'(\zeta)^2 - 48\gamma\Phi(\zeta)\Phi''(\zeta) + 7\gamma\zeta\Phi(\zeta) - 24\beta\Phi''''(\zeta) = 0. \tag{54}$$

Group-invariant solution via generator G_6

We consider Lie point symmetries $G_6 = 2\gamma F(t)\partial/\partial x + \alpha F'(t)\partial/\partial u$.

Solving

$$\frac{dt}{0} = \frac{dx}{2\gamma F(t)} = \frac{dy}{0} = \frac{du}{\alpha F'(t)}, \tag{55}$$

we achieve the group-invariant together with the involved invariants as

$$u = Q(T, Y) + \frac{\alpha F'(t)x}{2\gamma F(t)} \text{ with } T = t \text{ as well as } Y = y. \tag{56}$$

Inserting the expression of u in equation (5), then we have

$$\gamma Q_{YY} - \frac{\alpha^2 F''(t)x}{2\gamma F(t)} = 0. \tag{57}$$

Solving equation (57), we secure a solution of (5) in this regard as

$$u(t,x,y) = \frac{\alpha F'(t)x}{2\gamma F(t)} + \frac{\alpha^2 F''(t)}{4\gamma^2 F(t)} y^2 + f_1(t)y + f_2(t), \tag{58}$$

where arbitrary functions $f_1(t)$ and $f_2(t)$ are depending on t. Now, we want to contemplate some certain solutions of (5) by assigning specific functions to $F(t)$.

Case a. $F(t) = at^2 + bt + c$

Considering the quadratic function $F(t) = at^2 + bt + c$, we have

$$u = Q(T, Y) + \frac{\alpha x(2at + b)}{2\gamma(at^2 + bt + c)} \text{ with } T = t \text{ and } Y = y. \tag{59}$$

which is the group-invariant. Utilizing the function in (5), we achieve

$$a\gamma^2 X^2 Q_{YY} + b\gamma^2 X Q_{YY} + \gamma^2 c Q_{YY} - a\alpha^2 = 0. \tag{60}$$

The outcome of (60) gives an algebraic solution of (5) in this instance as

$$u(t, x, y) = \frac{a\alpha^2 y^2}{2(a\gamma^2 t^2 + b\gamma^2 t + c\gamma^2)} + \frac{\alpha x(2at + b)}{2\gamma(at^2 + bt + c)} + f_1(t)y + f_2(t). \tag{61}$$

We plot the algebraic function solution (61) taking $f_1(t) = \text{sech}(t)$ and $f_2(t) = \text{Si}(t)$ with the arbitrary choice of constants $a = 5$, $\alpha = 0.1$, $b = 1$. $\gamma = 2$, $c = 10$. Thus, we have the Figure

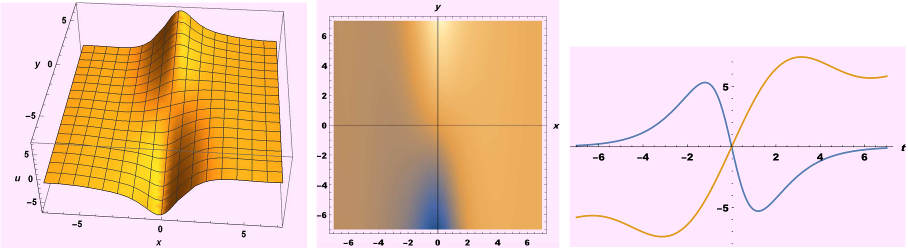

Figure: Wave depiction of algebraic function solution (61).

Case b. $F(t) = a\sin(t) + b\cos(t) + c$

We contemplate trigonometric function $F(t) = a\sin(t) + b\cos(t) + c$. In consequence, we secure a group-invariant of generator G_6 which is presented in this case as

$$u = Q(T, Y) + \frac{\alpha x \{a\cos(t) - b\sin(t)\}}{2\gamma\{a\sin(t) + b\cos(t) + c\}} \text{ where } T = t \text{ alongside } Y = y. \tag{62}$$

Engaging function u presented at (62) in (5), then we get equation

$$2a\gamma^2\sin(X)Q_{YY} + 2b\gamma^2\cos(X)Q_{YY} + a\alpha^2\sin(X) + \alpha^2 b\cos(X) + 2\gamma^2 cQ_{YY} = 0. \tag{63}$$

The solution to equation (63) produces a trigonometric result that satisfies (5) as

$$u(t,x,y) = \frac{\alpha x\left\{a\cos(t) - b\sin(t)\right\}}{2\gamma\left\{a\sin(t) + b\cos(t) + c\right\}} - \frac{\alpha^2\left\{b\cos(t) + a\sin(t)\right\}y^2}{4\left\{a\gamma^2\sin(t) + b\gamma^2\cos(t) + c\gamma^2\right\}} + f_1(t)y + f_2(t). \tag{64}$$

We make the same choice of the arbitrary functions as earlier revealed to represent trigonometric solution (64), occasioned by imploring a suitable selection of arbitrary constants $a = 3$, $\alpha = 0.1$, $b = 1$, $\gamma = 1$, $c = 10$. Therefore, we have the Figure

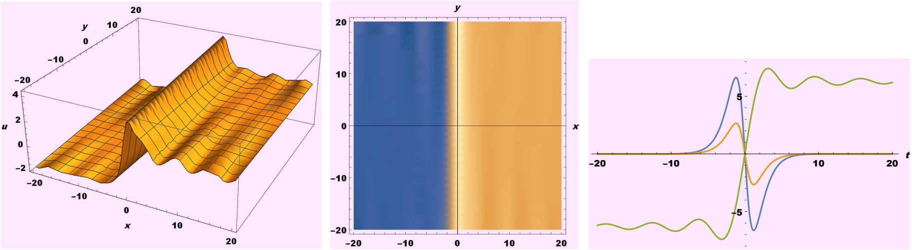

Figure: Wave depiction of trigonometric function solution (64).

Case c. $F(t) = a\,\mathrm{sech}(t) + b\cosh(t) + c$

Now, we consider hyperbolic function $F(t) = a\,\mathrm{sech}(t) + b\cosh(t) + c$. Similarity solution related to G_6 furnishes the function calculated as

$$u = Q(T, Y) + \frac{\alpha x \{b \sinh(3t) - 4a \sinh(t) + b \sinh(t)\}}{2\gamma \{2c \cosh(2t) + b \cosh(3t) + 4a \cosh(t) + 3b \cosh(t) + 2c\}} \tag{65}$$

where $T = t$ as well as $Y = y$. On using the obtained function u in (65) to replace that in (5), then we get a transformed structured of (5) as

$$\begin{aligned} &2b\gamma^2 \cosh(X)^4 Q_{YY} - \alpha^2 b \cosh(X)^4 + 2c\gamma^2 \cosh(X)^3 Q_{YY} \\ &+ 2a\gamma^2 \cosh(X)^2 Q_{YY} - a\alpha^2 \cosh(X)^2 + 2a\alpha^2 = 0. \end{aligned} \tag{66}$$

On solving equation (66), we gain hyperbolic function outcome satisfying (5) as

$$u(t,x,y) = \frac{\alpha x\left\{b\sinh(3t) - 4a\sinh(t) + b\sinh(t)\right\}}{2\gamma(2c\cosh(2t) + b\cosh(3t) + 4a\cosh(t) + 3b\cosh(t) + 2c)} + \frac{\left\{\alpha^2 b\cosh(t)^4 + a\alpha^2\cosh(t)^2 - 2a\alpha^2\right\}}{4\gamma^2\left\{b\cosh(t)^4 + c\cosh(t)^3 + a\cosh(t)^2\right\}}y^2 + f_1(t)y + f_2(t). \tag{67}$$

Remark

We notice that arbitrary function $F(t)$ in generator G_6 can assume various mathematical functions ranging from polynomial of any order, trigonometric to hyperbolic functions and give a solution of equation (5).

Here, we let $f_1(t) = f_2(t) = 0$ and see the resultant wave deflection. On imploring the constant values $a = 1$, $\alpha = 0.1$, $b = 1$, $\gamma = 0.1$, $c = 20$. We achieve a parabolic wave structure displayed as,

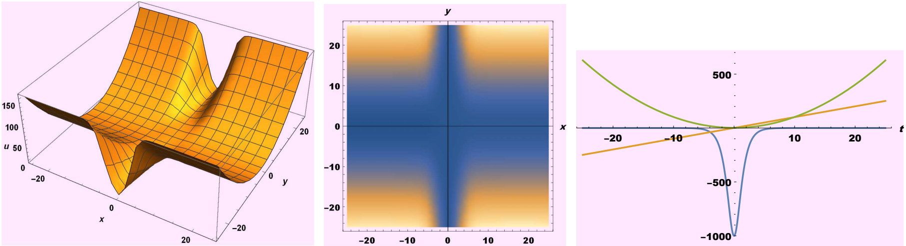

Figure: Wave depiction of hyperbolic function solution (67).

Conservation laws

In

P.J. Olver, Application of Lie Groups to Differential Equations, Springer, New York, 1993

the author gave a summary of the general algorithmic strategy to secure all conserved vectors accompanying differential equations including the one under study. Besides, they bequeathed a general exploration of a direct technique in gaining conserved vectors for differential equations. The reader can visit the references for better understanding.

In determining the conserved vectors of 2D-gNLWE (5), we aim at securing the zeroth-order multiplier and so the governing equation reads

$$\frac{\delta}{\delta u}[\Lambda\left(t,x,y,u\right)\left(\beta u_{xxxx}+\alpha u_{xt}+2\gamma u_x^2+2\gamma u u_{xx}-\gamma u_{yy}\right)]=0. \quad (68)$$

We define the Euler-Lagrange operator $\delta/\delta u$ in this regard as

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_x\frac{\partial}{\partial u_x}+D_x^2\frac{\partial}{\partial u_{xx}}+D_xD_t\frac{\partial}{\partial u_{xt}}+D_y^2\frac{\partial}{\partial u_{yy}}+D_x^4\frac{\partial}{\partial u_{xxxx}}, \quad (69)$$

where D_t, D_x and D_y are known as total derivatives just as defined in (10).

For 2D-gNLWE (5) a conserved vector can be demonstrated in a parallel structure by a divergence specification which is

$$D_t C^t + D_x C^x + D_y C^y = (\beta u_{xxxx} + \alpha u_{xt} + 2\gamma u_x^2 + 2\gamma u u_{xx} - \gamma u_{yy})\Lambda(t, x, y, u). \quad (70)$$

This specification is referred to as the characteristic equation for the prescribed conserved density C^t as well as spatial fluxes C^x and C^y.

Enlarging (68) and following the algorithm used in obtaining Lie point symmetries with the use of a Maple-based computer package,

A.F. Cheviakov, Computation of fluxes of conservation laws, J. Eng. math., 66 (2010) 153–173,

For 2D-gNLWE (5) a conserved vector can be demonstrated in a parallel structure by a divergence specification which is

$$D_t C^t + D_x C^x + D_y C^y = (\beta u_{xxxx} + \alpha u_{xt} + 2\gamma u_x^2 + 2\gamma u u_{xx} - \gamma u_{yy})\Lambda(t, x, y, u). \tag{70}$$

This specification is referred to as the characteristic equation for the prescribed conserved density C^t as well as spatial fluxes C^x and C^y. Enlarging (68) and following the algorithm used in obtaining Lie point symmetries with the use of a Maple-based computer package,

A.F. Cheviakov, Computation of fluxes of conservation laws, J. Eng. math., 66 (2010) 153–173, we get a system of PDE which solves to give the value of $\Lambda = \Lambda(t, x, y, u)$ as

$$\Lambda = \frac{1}{6\gamma}\left(\alpha y^3 f_4'(t) + 3\alpha y^2 f_3'(t) + 6\gamma(xyf_4(t) + xf_3(t)) + yf_2(t) + f_1(t)\right), \quad (71)$$

where arbitrary functions $f_4(t), f_3(t), f_2(t), f_1(t)$ are all depending on t.

Relative to the above multiplier we achieve the following conserved vectors of (5) preserved by the conserved densities alongside fluxes presented in the **Case a** to **Case d**. They are:

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Case a. $\Lambda_1 = f_1(t)$.

$$C_1^t = -\frac{1}{2}\alpha f_1(t) u_x,$$

$$C_1^x = \frac{1}{2}\left\{\alpha u f_1'(t) - \alpha u_t f_1(t) - 4\gamma u u_x f_1(t) - 2\beta u_{xxx} f_1(t)\right\},$$

$$C_1^y = \gamma u_y f_1(t);$$

Case b. $\Lambda_2 = y f_2(t)$.

$$C_2^t = -\frac{1}{2}\alpha y u_x f_2(t),$$

$$C_2^x = -\frac{1}{2}y\left\{4\gamma u u_x f_2(t) - \alpha u f_2'(t) + \alpha u_t f_2(t) + 2\beta u_{xxx} f_2(t)\right\},$$

$$C_2^y = -\gamma f_2(t)\Big(u - y u_y\Big);$$

Case c. $\Lambda_3 = \frac{1}{6\gamma}\left(\alpha y^3 f_3'(t) + 6\gamma xyf_3(t)\right)$

$$C_3^t = \frac{\alpha}{2\gamma} y \left\{ \gamma(u - xu_x)f_3(t) - \frac{\alpha}{6} y^2 u_x f_3'(t) \right\},$$

$$C_3^x = -\frac{1}{12\gamma} y \left\{ -\alpha^2 y^2 u f_3''(t) - 6\alpha \left[\gamma u \left(x - \frac{2}{3} y^2 u_x \right) - \frac{1}{6} y^2 (\alpha u_t \right.\right.$$

$$\left. + 2\beta u_{xxx}) \right] f_3'(t) + 6\gamma(2\gamma(2xuu_x - u^2) + \alpha x u_t$$

$$\left. + 2\beta(xu_{xxx} - u_{xx}))f_3(t) \right\},$$

$$C_3^y = -\frac{1}{4} \left\{ 2\alpha y^2 \left(u - \frac{1}{3} y u_y \right) f_3'(t) + 4\gamma x(u - yu_y) f_3(t) \right\};$$

Case d. $\Lambda_4 = \frac{1}{\gamma}\left(\frac{\alpha}{2}y^2 f_4'(t) + \gamma x f_4(t)\right)$

$$C_4^t = \frac{\alpha}{2\gamma}\left\{\gamma u f_4(t) - \frac{1}{2}\alpha y^2 u_x f_4'(t) - \gamma x u_x f_4(t)\right\},$$

$$C_4^x = \frac{1}{4\gamma}\left\{\alpha^2 y^2 u f_4''(t) - \alpha(-2\gamma u(x - 2y^2 u_x) + y^2(\alpha u_t + 2\beta u_{xxx}))f_4'(t) - 2\gamma(2\gamma(2xuu_x - u^2) + \alpha x u_t + 2\beta(x u_{xxx} - u_{xx}))f_4(t)\right\},$$

$$C_4^y = -\frac{1}{2}\left\{\alpha y(2u - y u_y) f_4'(t) - 2\gamma x u_y f_4(t)\right\}.$$

Concluding remarks

Conclusion

In this talk we presented the solutions of the 2D-gNLWE (5). On imploring the Lie group technique, we have been able to successfully obtain the associated group-invariant solutions to the generated point symmetries of (5) through the reduction process.

A unique observation from an arbitrary-function-containing-symmetry of a nonlinear partial differential equation was presented in this study, whereby upon reduction, the arbitrary functions involved, assumed various mathematical functions whose final solutions satisfy (5). The obtained

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A unique observation from an arbitrary-function-containing-symmetry of a nonlinear partial differential equation was presented in this study, whereby upon reduction, the arbitrary functions involved, assumed various mathematical functions whose final solutions satisfy (5). The obtained solutions are highly important. They contain mathematical functions with the inclusion of elliptic (Jacobi and Weierstrass), trigonometric, hyperbolic, algebraic along with solutions with arbitrary functions.

Besides, we established a known fact that considering the special limits of a cnoidal solution, it disintegrates to elementary functions. Several of the solution achieved, appear as solitons of different kinds ranging from trigonometric, non-topological 1-soliton, complex, dark and bright to topological kink soliton. Finally, the conserved quantities of (5) are established by utilizing the general multiplier technique.

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Thank you so much for your attention