A FAST AND SIMPLE METHOD FOR SOLVING QUASIMONOTONE VARIATIONAL INEQUALITIES

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Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let $A: H \to H$ be a single valued mapping. Let $\langle . \rangle$ and $\| . \|$ denote the inner product and induced norm on H, respectively. The classical variational inequality problem (VIP), introduced by Fichera [2] and Stampacchia [8] is defined as finding a point $x^* \in C$, such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \qquad \forall x \in C.$$

We denote the solution set of the VIP by V_I. Over the years, variational inequality theory has proven to be a major area of research in mathematical analysis, and has drawn the attention of several researchers due to its wide range of applications in diverse fields, such as optimal control, game theory, signal processing, linear programming, image recovery, etc. (See [3,5]).

We define the dual variational inequality problem (DVIP) as finding a point $x^* \in C$, such that

$$\langle Ax, x - x^* \rangle \geq 0, \qquad \forall x \in C.$$

We denote the solution set of the VIP by V_D. Note that If A is quasimonotone and continous, then $V_I \nsubseteq V_D$. Although, $V_D \subseteq V_I$. Note that A is quasimonotone if

$$\langle Ay, x - y \rangle > 0 \Rightarrow, \langle Ax, x - y \rangle \geq 0, \qquad \forall x, y \in H.$$

The class of quasimonotone variational inequality problems is known to be more general and more applicable than the classes of pseudomonotone and monotone variational inequality problems. Unfortunately, there are only very few results in the literature on quasimonotone variational inequality problems, and several of these results only proves a weak convergence result. However, in this research, we prove a strong convergence result, which is far better [1]. We also note a very vital tool for expediting the rate of convergence of our method, Known as the **inertial iteration** [7].

Literature Review

Recently, Liu and Yang [6] proposed a new self-adaptive method for solving the variational inequalities with quasimonotone operator. Yin et al. [10] proposed the following iterative algorithm for approximating the common solution of fixed point problem and quasimonotone variational inequalities. Very recently, Yin and Hussain [9] proposed a forward-backward-forward algorithm for solving quasimonotone variational inequalities. Also, Izuchukwu et al. [4] proposed a new projection method for solving quasimonotone variational inequality problems.

We note that the authors [4,6,9,10] only succeeded in proving a weak convergence theorem. Moreso, their results were achieved under some very strict conditions, like the Lipschitz continuity, sequentially weakly continuity, and a finite non-trivial solution set.

In this work, we adopt an inertial and self-adaptive method, and we prove a strong convergence theorem, under more relaxed conditions.

Algorithm (Alg. 3.1)

tep 0. Let $x_0, x_1 \in H$ be chosen arbitrarily and set $n = 1$.

tep 1. choose δ_n such that $0 \leq \delta_n \leq \hat{\delta}_n$ with $\hat{\delta}_n$ defined by

$$\hat{\delta}_n = \begin{cases} \min\left\{\delta, \ \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \delta, & \text{otherwise.} \end{cases}$$

tep 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}); \quad y_n = P_C(w_n - \gamma_n A w_n).$$

tep 3. Compute $z_n = w_n - l\tau_n d_n$, where,

$$d_n := w_n - y_n - \gamma_n(Aw_n - Ay_n),$$

$$\tau_n = \frac{\langle w_n - y_n, d_n \rangle}{\|d_n\|^2}, \quad d_n \neq 0.$$

$$\gamma_{n+1} = \begin{cases} \min\{\frac{\sigma\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \ \gamma_n + \theta_n\}, & \text{if } Aw_n \neq Ay_n, \\ \gamma_n + \theta_n, & \text{otherwise.} \end{cases}$$

tep 4. Compute $x_{n+1} = (1 - \beta_n)(\alpha_n w_n) + \beta_n z_n$. Set $n := n + 1$

Results

In this paper, we achieve a strong convergence result as follows:

THEOREM: Let $\{x_n\}$ be the sequence generated by our algorithm (Alg. 3.1), which satisfies some given mild conditions, and $Ax \neq 0, \ \forall x \in C$. Then, $\{x_n\}$ converges strongly an element $x^* \in V_D \subset V_I$, which is a minimum–norm solution of the problem, where $\|x^*\| = \min\{\|p\|: p \in V_D \subset V_I\}$.

The proof of our strong convergence theorem is clearly shown in our main paper. We proceed to display some numerical experiments to show the performance of our method in comparison with other methods in literature.

Table 1. Experiment 1 Table.

Algorithms compared	CPU time (sec.)	No. of Iterations
Alg. 1.2 [6]	0.0138	112
Alg. 1.5 [8]	0.0048	41
Alg. 1.6 [7]	0.0045	112
App 6.1 [4]	0.0073	150
App 6.2 [1]	0.0054	121
App 6.3 [1]	0.0051	121
Our Algorithm (Alg 3.1)	0.0069	27

Table 2. Experiment 2 Table.

Algorithms compared	CPU time (sec.)	No. of Iterations
App 6.2 [1]	0.0152	49
App 6.3 [1]	0.0059	49
Our Algorithm (Alg 3.1)	0.0089	31

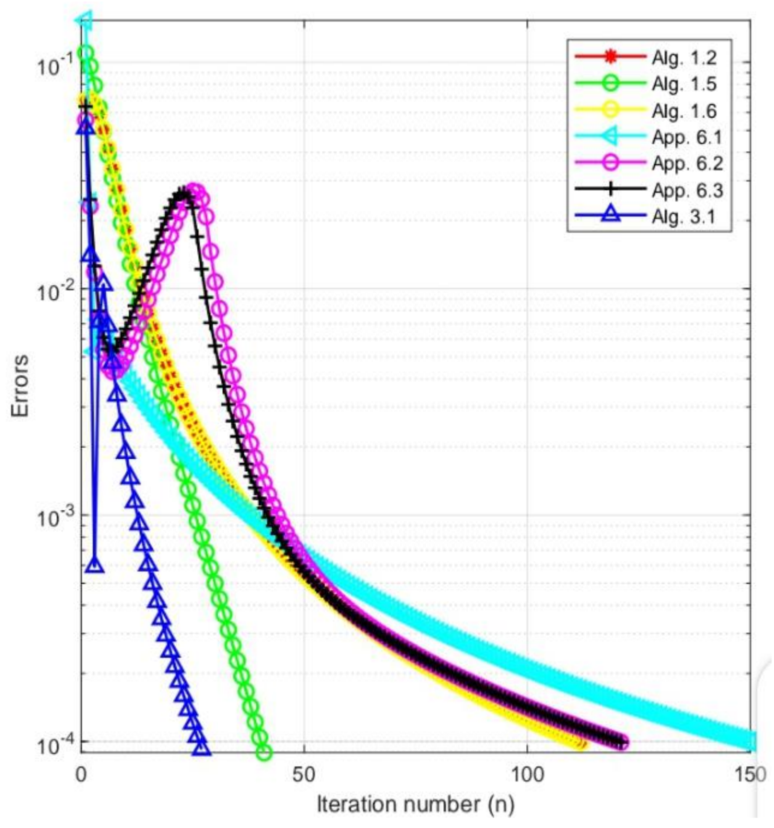

Figure 1. Experiment 1 plot.

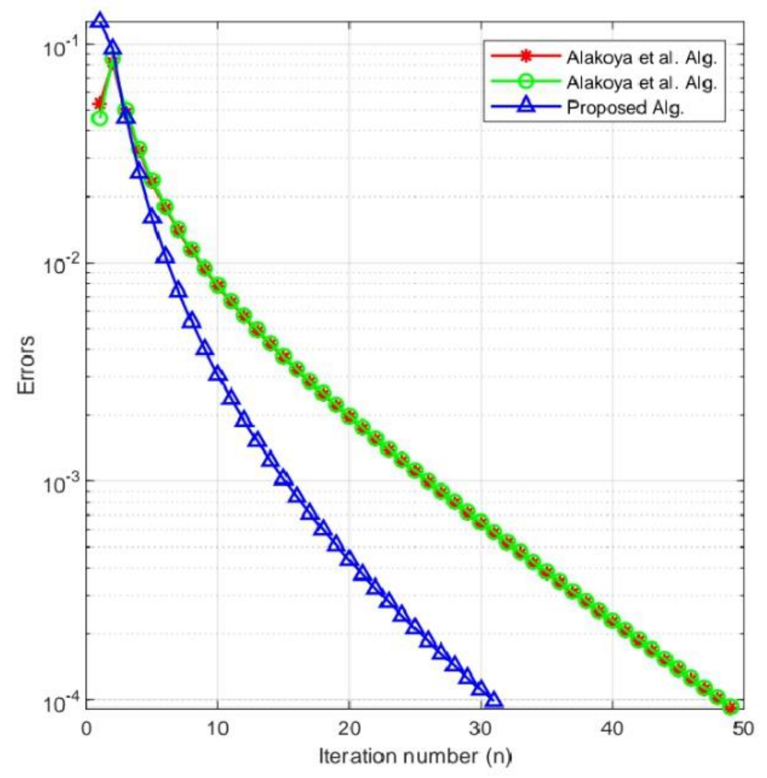

Figure 2. Experiment 2 plot.

Discussion

In our experiment 1&2, we compare our Algorithm (Alg. 3.1) with some other methods in literature, and we plot the graph of errors against number of iterations, as shown in Figures 1 and 2. We observe from Table 1 and 2, that our Algorithm (Alg. 3.1) has the least number of iterations, and thus converges faster than the other algorithms in literature.

Clearly, our method is more efficient and performs better than all the other algorithms compared in literature.

Conclusion

In this paper, we studied the class of quasimonotone variational inequality problems in real Hilbert spaces. We proposed a new Mann-type inertial projection and contraction method with self-adaptive step size. Our proposed method does not require the associated cost operator to be Lipschitz continuous and does not involve any linesearch technique. We showed that our method converges strongly to a minimum-norm solution of the problem in consideration, under more relaxed conditions. Finally, we presented some numerical experiments to show that our algorithm performs better than some existing methods in literature.

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