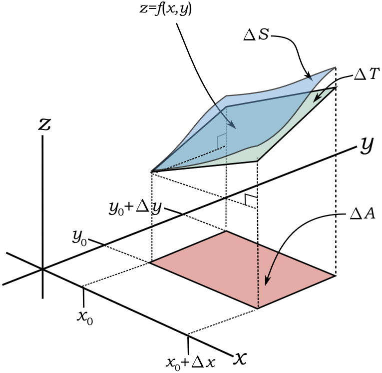
[9.13] SURFACE INTEGRALS, DERIVATION OF SURFACE ELEMENT A surface integral:

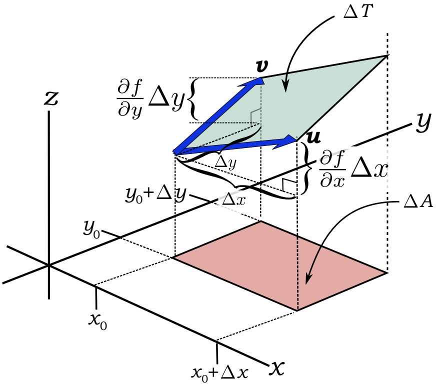
$$\iint_S G(x,y,z)\,dS$$

We shall derive the formula for the surface element:

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,dA$$



- ΔS is a surface element (part of the surface S) with one corner at (x_0, y_0).
- ΔT is a parallelogram with one corner at (x_0, y_0), that has the same slope as ΔS at (x_0, y_0).
- ΔA is the projection of ΔS or ΔT on the xy-plane.



- **u** and **v** are the vectors along the two sides of ΔT positioned at $(x_0, y_0, f(x_0, y_0))$.
- The area of such a parallelogram is: area $= \|\mathbf{u} \times \mathbf{v}\|$
- $\mathbf{u} = \begin{bmatrix} \Delta x \\ 0 \\ \frac{\partial f}{\partial x}\Delta x \end{bmatrix}$
- $\mathbf{v} = \begin{bmatrix} 0 \\ \Delta y \\ \frac{\partial f}{\partial y}\Delta y \end{bmatrix}$

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} \Delta x \\ 0 \\ f_x\ \Delta x \end{bmatrix} \times \begin{bmatrix} 0 \\ \Delta y \\ f_y\ \Delta y \end{bmatrix} = \begin{bmatrix} -f_x \Delta x \Delta y \\ -f_y \Delta x \Delta y \\ \Delta x \Delta y \end{bmatrix} = \Delta x \Delta y \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}$$

$$\Delta T = \|\mathbf{u} \times \mathbf{v}\| = \left\| \Delta x \Delta y \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix} \right\|$$

$$= \Delta x \Delta y \left\| \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix} \right\|$$

$$= \Delta x \Delta y \sqrt{(-f_x)^2 + (-f_y)^2 + 1^2}$$

$$= \sqrt{1 + (f_x)^2 + (f_y)^2} \;\; \Delta A$$

As $\Delta x \to 0$ and $\Delta y \to 0$,

$$\Delta S \to \Delta T$$

and we can replace "Δ" with "d".

$$dS = \sqrt{1 + (f_x)^2 + (f_y)^2} \;\; dA$$