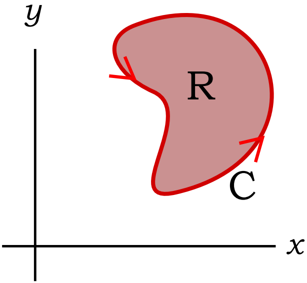
The three theorems:

- **Green:** Closed line integral (in xy-plane) $\leftrightarrow$ Double integral (in xy-plane)
- **Stokes:** Closed line integral (in 3D space) $\leftrightarrow$ Surface integral (in 3D space)
- **Gauss/Divergence:** Closed surface integral (in 3D space) $\leftrightarrow$ Volume integral (in 3D space)

Green's Theorem:

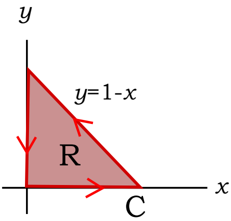


$$\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Example: Verify Green's theorem (i.e. do both sides and check)

$$W = \oint_C \mathbf{F} \cdot d\mathbf{r}, \qquad \mathbf{F} = \begin{bmatrix} xy \\ 3x \end{bmatrix}$$



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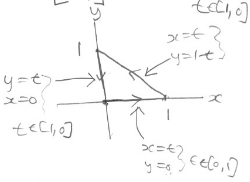
Example 1: Verify Green.

$$W = \oint_C \underline{F}\cdot d\underline{r} \qquad \underline{F} = \begin{bmatrix} xy \\ 3x \end{bmatrix}$$

LHS

$$W = \oint_C xy\,dx + 3x\,dy$$



$$W = \int_0^1 0 + 0 + \int_1^0 t(1-t)\,dt + 3t(-dt) + \int_1^0 0$$

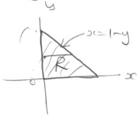
$$= \int_1^0 (-t^2 + t - 3t)\,dt = \int_1^0 (-2t - t^2)\,dt$$

$$= \frac{4}{3}$$

RHS: $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = \dfrac{\partial}{\partial x}(3x) - \dfrac{\partial}{\partial y}(xy)$

$$= 3 - x$$

$$W = \iint_R (3-x)\, dA$$



$$= \int_{y=0}^{1} \int_{x=0}^{1-y} (3-x)\, dx\, dy$$

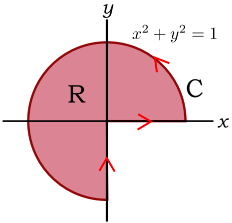
$$= \int_0^1 \left[3x - \frac{x^2}{2} \right]_0^{(1-y)} dy$$

$$= \int_0^1 \left(3(1-y) - \tfrac{1}{2}(1-y)^2 \right) dy$$

$$= \frac{4}{3}$$

Example 2: Use Green's Theorem to calculate

$$I = \oint_C (x - x^2)y \, dx + xy^2 \, dy$$



Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

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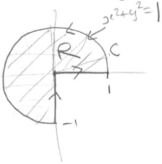
Example 2: Use Green to calculate

$$I = \oint_C (x - x^2)y\,dx + xy^2\,dy$$

Do the RHS:



- $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = y^2 - (x - x^2)$
  $= x^2 + y^2 - x$

$$I = \iint_R \overbrace{(x^2+y^2-x)}\,dA$$

$x = r\cos\theta$
$y = r\sin\theta$

Do in polar.

$r\,dr\,d\theta$

describe R i.e. r and θ.

$$I = \int_{\theta=0}^{\frac{3\pi}{2}} \int_{r=0}^{1} \left(r^2 - r\cos\theta\right) r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\frac{3\pi}{2}} \left[\int_{r=0}^{1} \left(r^3 - r^2\cos\theta\right) dr \right] d\theta$$

$$= \int_0^{\frac{3\pi}{2}} \left[\frac{r^4}{4} - \frac{r^3}{3} \cos\theta \right]_0^1 d\theta \qquad \boxed{5}$$

$$= \int_0^{3\pi/2} \left(\frac{1}{4} - \frac{1}{3} \cos\theta \right) d\theta$$

$$= \frac{3\pi}{8} + \frac{1}{3}$$