STELLENBOSCH UNIVERSITY

APPLIED MATHEMATICS 20753-242

VECTOR-ANALYSIS

NOTES ON FINDING POTENTIAL FUNCTIONS

If a vector field is conservative ($\equiv$ irrotational $\equiv$ a gradient field), line integrals of the form

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}, \tag{1}$$

only depend on the starting point and end point of the path C and not on the specific path that joins the two points (provided that no path runs through a singularity in $\mathbf{F}$, or encircles a singularity in $\mathbf{F}$).

The integral in (1) is then simply written as

$$W = \int_A^B \mathbf{F} \cdot d\mathbf{r}, \tag{2}$$

where A and B are labels for the starting point and end point of the path C.

In order to test whether $\mathbf{F}$ is irrotational, one simply calculates $\nabla \times \mathbf{F}$ and check that it is zero. If it is, then $\mathbf{F}$ is the gradient of some function $\phi(x, y, z)$ called the potential function of $\mathbf{F}$. Let

$$\mathbf{F} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

This means that

$$\mathbf{F} = \nabla\phi = \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \\ \frac{\partial \phi}{\partial y} \\ \\ \frac{\partial \phi}{\partial z} \end{bmatrix}.$$

In other words,

$$P = \frac{\partial \phi}{\partial x}, \quad , \quad Q = \frac{\partial \phi}{\partial y}, \quad \text{and} \quad R = \frac{\partial \phi}{\partial z}. \tag{3}$$

From the chain rule for partial derivatives we have that

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz, \tag{4}$$

but notice that $\mathbf{F} \cdot d\mathbf{r}$ is expanded as follows

$$\mathbf{F} \cdot d\mathbf{r} = Pdx + Qdy + Rdz, \tag{5}$$

and by comparing (5) to (4), it is clear that

$$\mathbf{F} \cdot d\mathbf{r} = d\phi. \tag{6}$$

The integral (2) can then be integrated simply as follows:

$$W = \int_A^B d\phi = \Big[\phi(x, y, z)\Big]_A^B. \tag{7}$$

How can one find ϕ from $\mathbf{F}$? We show two different ways here, which we shall simply call [1] the Integration-Differentiation method, and [2] the Term-Collection method. These methods will be illustrated by means of an example.

Example: Find the potential function of $\mathbf{F} = (2xyz^3+3z)\mathbf{i}+(x^2z^3+z)\mathbf{j}+(3x^2yz^2+3x+y+5)\mathbf{k}$.

Let us first check that the vector field is conservative.

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} 2xyz^3 + 3z \\ x^2z^3 + z \\ 3x^2yz^2 + 3x + y + 5 \end{bmatrix} = \begin{bmatrix} (3x^2z^2 + 1) - (3x^2z^2 + 1) \\ (6xyz^2 + 3) - (6xyz^2 + 3) \\ 2xz^3 - 2xz^3 \end{bmatrix} = \mathbf{0}$$

Yes, it is.

Finding ϕ using the Integration-Differentiation Method:

Since

$$P = \frac{\partial \phi}{\partial x}, \qquad Q = \frac{\partial \phi}{\partial y}, \qquad \text{and} \qquad R = \frac{\partial \phi}{\partial z}, \tag{8}$$

we may integrate the first equation of (8) to x,

$$\begin{aligned} \int d\phi &= \int (2xyz^3 + 3z)dx \\ \phi &= x^2yz^3 + 3xz + g(y, z) \end{aligned} \tag{9}$$

where $g(y, z)$ is an 'integration constant' here, i.e. it is constant with respect to x, but may depend on y and z.

In order to find $g(y, z)$ we differentiate (9) to y, and compare it to Q,

$$\frac{\partial \phi}{\partial y} = x^2z^3 + \frac{\partial g}{\partial y} = x^2z^3 + z, \tag{10}$$

therefore $\frac{\partial g}{\partial y} = z$. Integrating this to y gives

$$\int \frac{\partial g}{\partial y} = \int z dy, \tag{11}$$

and therefore

$$g(y, z) = yz + h(z), \tag{12}$$

where $h(z)$ is an integration constant with respect to y, that may depend on z. Substituting (12) into (9), gives

$$\phi = x^2yz^3 + 3xz + yz + h(z). \tag{13}$$

Lastly one must integrate (13) to z, and compare it to R in order to obtain the correct expression for $h(z)$.

$$\frac{\partial \phi}{\partial z} = 3x^2yz^2 + 3x + y + h'(z) = 3x^2yz^2 + 3x + y + 5,$$

therefore $h'(z) = 5$, and integration to z gives

$$h(z) = 5z + C. \tag{14}$$

After substituting (14) into (13), we obtain

$$\phi = x^2yz^3 + 3xz + yz + 5z + C. \tag{15}$$

<u>Finding ϕ using the Term-collection Method:</u>

This method appears to be shorter, but one must be careful that you collect the terms correctly.

Once again we make use of the fact that

$$P = \frac{\partial \phi}{\partial x}, \quad , \quad Q = \frac{\partial \phi}{\partial y}, \quad \text{and} \quad R = \frac{\partial \phi}{\partial z}. \tag{16}$$

We integrate P to x, Q to y and R to z:

$$\begin{aligned} \int d\phi &= \int P dx \\ \phi &= \int (2xyz^3 + 3z) dx \\ &= x^2yz^3 + 3xz + \underset{\substack{\uparrow \\ \text{terms in} \\ y \text{ and } z}}{\ldots\ldots\ldots} \end{aligned}$$

$$\begin{aligned} \int d\phi &= \int Q dy \\ \phi &= \int (x^2z^3 + z) dy \\ &= x^2yz^3 + yz + \underset{\substack{\uparrow \\ \text{terms in} \\ x \text{ and } z}}{\ldots\ldots\ldots} \end{aligned}$$

$$\begin{aligned} \int d\phi &= \int R dz \\ \phi &= (3x^2yz^2 + 3x + y + 5) dz \\ &= x^2yz^3 + 3xz + yz + 5z + \underset{\substack{\uparrow \\ \text{terms in} \\ x \text{ and } y}}{\ldots\ldots\ldots} \end{aligned}$$

In each of these expressions there are terms missing (the terms depending on the other two variables) when one integrates to a particular variable. Of course, any term that depends on x, y, as well as z, will appear in all three cases. A term in x and z for example, (such as the term $3xz$ in the example) will appear when one has integrated to x as well as to z but it will not appear when one integrates to y. Is it then necessary to collect the terms correctly to form the function ϕ. In this example, therefore,

$$\phi = x^2yz^3 + 3xz + yz + 5z + C, \tag{17}$$

where we have added the constant C.