[9.16] 2, 3, 5, 8, 13, 21.

[9.16] 02a.

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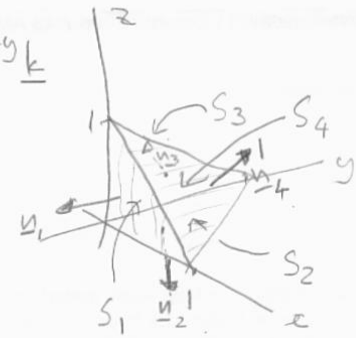
2. $\underline{E} = 6xy\,\underline{i} + 4yz\,\underline{j} + xe^{-y}\,\underline{k}$



$$\iint_S \underline{E}\cdot\underline{n}\,dS$$

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4}$$

$$= \iint_{S_1} \underline{E}\cdot(-\underline{j})\,dS + \iint_{S_2} \underline{E}\cdot(-\underline{k})\,dS + \iint_{S_3} \underline{E}\cdot(-\underline{i})\,dS$$

$$+ \iint_{S_4} \underline{E}\cdot\frac{1}{\sqrt{3}}(\underline{i}+\underline{j}+\underline{k})\,dS$$

$$= \iint_{S_1} -4yz\,dS + \iint_{S_2} -xe^{-y}\,dS + \iint_{S_3} -6xy\,dS +$$

$$+ \iint_{S_4} \frac{1}{\sqrt{3}}\left[6xy + 4yz + xe^{-y}\right] dS$$

$$= \int_{x=0}^{1}\int_{z=0}^{1-x} -4\underset{\substack{\uparrow\\ 0}}{y}z\,dz\,dx + \int_{x=0}^{1}\int_{y=0}^{1-x} -xe^{-y}\,dy\,dx$$

$$+ \int_{y=0}^{1}\int_{z=0}^{1-y} -6\underset{\substack{\uparrow\\ 0}}{x}y\,dz\,dy + \int_{x=0}^{1}\int_{y=0}^{1-x} \left(6xy + 4y\underset{\substack{\uparrow\\ (1-x-y)}}{z} + xe^{-y}\right)\frac{1}{\sqrt{3}}\sqrt{3}\,dy\,dx$$

$$= 0 + \int_{x=0}^{1}\int_{y=0}^{1-x} -xe^{-y}\,dy\,dx + 0$$

$$+ \int_{x=0}^{1}\int_{y=0}^{1-x} \left(6xy + 4y(1-x-y) + xe^{-y}\right) dy\,dx$$

[9.16] 02b.

2 continued :

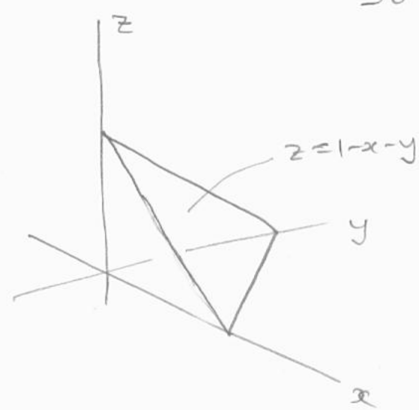
$$\iint_S \underline{F}\cdot\underline{n}\,dS = \int_{x=0}^{1}\int_{y=0}^{1-x}\left(6xy+4y-4xy-4y^2 \cancel{+xe^{-y}} \cancel{-xe^{-y}}\right)dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left(2xy+4y-4y^2\right)dy\,dx = \int_0^1\left[xy^2+\frac{4y^2}{2}-\frac{4y^3}{3}\right]_0^{1-x}dx$$

$$= \int_0^1\left(x(1-x)^2+2(1-x)^2-\frac{4}{3}(1-x)^3-0\right)dx$$

$$= \int_0^1\left(\frac{2}{3}+x-4x^2+\frac{7}{3}x^3\right)dx = \left[\frac{2}{3}x+\frac{x^2}{2}-\frac{4x^3}{3}+\frac{7}{3}\frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{3}+\frac{1}{2}-\frac{4}{3}+\frac{7}{12} = \frac{5}{12}$$



$$\iiint_D \underline{\nabla}\cdot\underline{F}\,dV = \iiint_D (6y+4z)\,dV$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}(6y+4z)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left[6yz+2z^2\right]_0^{1-x-y}dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left[6y(1-x-y)+2\left(1+x^2+y^2-2x-2y+2xy\right)\right]dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left(6y-6xy-6y^2+2+2x^2+2y^2-4x-4y+4xy\right)dy\,dx$$

[9.16] 02c.

2 continued further:

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \left(-4y^2 + 2x^2 - 2xy + 2y - 4x + 2\right) dy\, dx$$

$$= \int_{x=0}^{1} \left[-4\frac{y^3}{3} + 2x^2 y - \frac{\cancel{2}xy^2}{\cancel{2}} + \frac{\cancel{2}y^2}{\cancel{2}} - 4xy + 2y \right]_0^{1-x} dx$$

$$= \int_{x=0}^{1} \Big(-\frac{4}{3}(1-x)^3 + 2x^2(1-x) - x(1-x)^2 + (1-x)^2$$
$$- 4x(1-x) + 2(1-x) \Big) dx$$

$$= \int_{x=0}^{1} \Big(-\frac{4}{3}\left(1 - 3x + 3x^2 - x^3\right) + 2x^2 - 2x^3 - x\left(1 - 2x + x^2\right)$$
$$+ 1 - 2x + x^2 - 4x + 4x^2 + 2 - 2x \Big) dx$$

$$= \int_{x=0}^{1} \Big(-\frac{4}{3} + 4x - 4x^2 + \frac{4}{3}x^3 + 2x^2 - 2x^3 - x + 2x^2 - x^3$$
$$+ 1 - 2x + x^2 - 4x + 4x^2 + 2 - 2x \Big) dx$$

$$= \int_0^1 \left[\frac{5}{3} - 5x + 5x^2 - \frac{5}{3}x^3 \right] dx$$

$$= \left[\frac{5}{3}x - \frac{5}{2}x^2 + \frac{5}{3}x^3 - \frac{5}{3 \cdot 4}x^4 \right]_0^1$$

$$= \frac{5}{3} - \frac{5}{2} + \frac{5}{3} - \frac{5}{12} = 5\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \frac{1}{12} \right) = \frac{5}{12}$$

[9.16] 03.

3. $\iint_S (\underline{F}\cdot\underline{n})\,dS = \iiint_D \underline{\nabla}\cdot\underline{F}\,dV \qquad \underline{\nabla}\cdot\underline{F} = 3x^2+3y^2+3z^2$

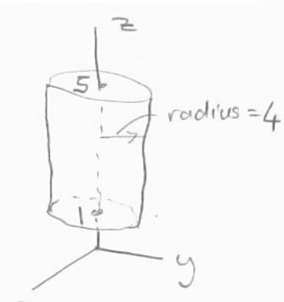
$$= \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi}\int_{r=0}^{a} 3r^2 \; r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$

$$= 3\int_0^{2\pi} d\theta \times \int_{\varphi=0}^{\pi} \sin\varphi\, d\varphi \times \int_{r=0}^{a} r^4\, dr$$

$$= 3\,(2\pi)(2) \times \left[\frac{r^5}{5}\right]_0^a = \frac{12}{5}a^5\pi$$

[9.16] 05.

5. $\underline{\nabla}\cdot\underline{F} = 0 + 0 + 2(z-1)$



$$\iint_S (\underline{F}\cdot\underline{n})\,dS = \iiint_D \underline{\nabla}\cdot\underline{F}\,dV.$$

$$= 2\int_{\theta=0}^{2\pi}\int_{r=0}^{4}\int_{z=1}^{5} (z-1)\, r\, dz\, dr\, d\theta$$

$$= 2\int_0^{2\pi} d\theta \times \int_0^4 r\, dr \times \int_1^5 (z-1)\,dz$$

$$= 2(2\pi)\left[\frac{r^2}{2}\right]_0^4 \left[\frac{z^2}{2} - z\right]_1^5 = 4\pi\left[\frac{16}{2}\right]\left[\frac{25}{2} - 5 - \frac{1}{2} + 1\right]$$

$$= 4\pi(8)(8) = 256\pi$$

[9.16] 08.

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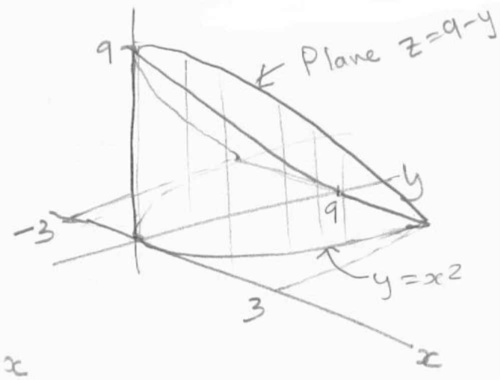
8. $\underline{\nabla} \cdot \underline{F} = 2x + 0 + 0$



$$\iint_S \underline{F} \cdot \underline{n}\, dS = \iiint_D (\underline{\nabla} \cdot \underline{F})\, dV$$

$$= \int_{x=-3}^{3} \int_{y=x^2}^{9} \int_{z=0}^{9-y} 2x \, dz\, dy\, dx$$

$$= \int_{x=-3}^{3} \int_{y=x^2}^{9} \Big[2xz \Big]_0^{9-y} dy\, dx = \int_{x=-3}^{3} \int_{y=x^2}^{9} 2x(9-y)\, dy\, dx$$

$$= \int_{x=-3}^{3} \int_{y=x^2}^{9} (18x - 2xy)\, dy\, dx = \int_{-3}^{3} \left[18xy - \frac{\cancel{2}xy^2}{\cancel{2}} \right]_{x^2}^{9} dx$$

$$= \int_{-3}^{3} \left(18x \cdot 9 - x\, 81 - 18x x^2 + x(x^2)^2 \right) dx$$

$$= \int_{-3}^{3} \left(162x - 81x - 18x^3 + x^5 \right) dx$$

$$= \int_{-3}^{3} \left(81x - 18x^3 + x^5 \right) dx = \left[\frac{81x^2}{2} - \frac{18x^4}{4} + \frac{x^6}{6} \right]_{-3}^{3}$$

$$= 0$$

[9.16] 13.

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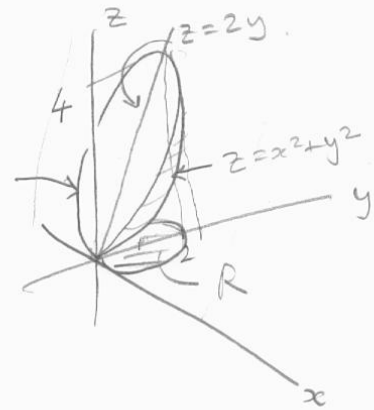
13. $\underline{\nabla}\cdot\underline{F} = 6xy^2 + 1 - 6xy^2$

$= 1$

$$\iint_S \underline{F}\cdot\underline{n}\,dS = \iiint_D \underline{\nabla}\cdot\underline{F}\,dV = \iiint_D 1\cdot dV$$



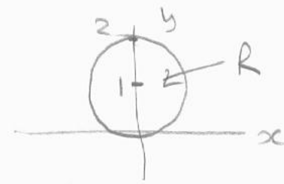
Region R:

$$2y = x^2 + y^2$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + (y-1)^2 = 1 \leftarrow$$ circle with centre at $(0,1)$ and radius 1.

$$\iiint_D dV = \iint_R \int_{z=x^2+y^2}^{2y} dz\,dA$$



$$= \iint_R \left[z\right]_{x^2+y^2}^{2y} dA = \iint_R (2y - x^2 - y^2)\,dA$$

Substitute $u = y - 1$ or $y = u + 1$, P is the circle with centre at origin.

$$= \iint_P \left(2(u+1) - x^2 - (u+1)^2\right) dA$$

$$= \iint_P (\cancel{2u} + 2 - x^2 - u^2 - \cancel{2u} - 1)\,dA$$

$$= \iint_P (1 - (x^2 + u^2))\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (1 - r^2)\,r\,dr\,d\theta$$

$$= 2\pi\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{4}\right) = \frac{\pi}{2}$$

[9.16] 21.

21. Let $\underline{a}$ be a constant vector, and

$\underline{F} = f\underline{a}$.

changes with x, y, z

$$\iint_S \underline{F}\cdot\underline{n}\, dS = \iint_S f\, \underline{a}\cdot\underline{n}\, dS$$

constant

$$= \underline{a}\cdot\iint_S f\, \underline{n}\, dS \quad \text{——} \quad (1)$$

$$\iiint_D (\underline{\nabla}\cdot\underline{F})\, dV = \iiint_D \underline{\nabla}\cdot(f\underline{a})\, dV$$

$$= \iiint_D \Big(\underline{\nabla} f\cdot\underline{a} + f\underbrace{(\underline{\nabla}\cdot\underline{a})}_{=0}\Big) dV.$$

$$= \iiint_D (\underline{a}\cdot\underline{\nabla} f)\, dV$$

constant

$$= \underline{a}\cdot\iiint_D \underline{\nabla} f\, dV \quad \text{——} \quad (2)$$

Let $\underline{a} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, then the first component of (1) is equal to the first component of (2).

Then let $\underline{a} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, i.e. second components are equal, ...etc.