TUT 11: [9.15] 2, 4, 11, 17, 21.51, 54, 55, 75, 78, 80.

[9.15] 02.

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2. $\displaystyle\int_{x=1}^{3}\int_{y=1}^{x}\int_{z=2}^{xy} 24xy\,dz\,dy\,dx$

$\displaystyle = \int_{x=1}^{3}\int_{y=1}^{x} 24xy\left[xy-2\right]dy\,dx$

$\displaystyle = \int_{x=1}^{3}\int_{y=1}^{x}\left(24x^2y^2-48xy\right)dy\,dx$

$\displaystyle = \int_{x=1}^{3}\left[\frac{24x^2y^3}{3}-\frac{48xy^2}{2}\right]_1^x dx$

$\displaystyle = \int_{x=1}^{3}\left(8x^5-24x^3-8x^2+24x\right)dx$

$\displaystyle = \left[\frac{8x^6}{6}-\frac{24x^4}{4}-\frac{8x^3}{3}+\frac{24x^2}{2}\right]_1^3$

$\displaystyle = \frac{8\times729}{6}-\frac{24\times81}{4}-\frac{8\times27}{3}+\frac{24\times9}{2}$

$\displaystyle \quad -\frac{8}{6}+\frac{24}{4}+\frac{8}{3}-\frac{24}{2}$

$\displaystyle = 522-\frac{14}{3} = \frac{1552}{3}$

[9.15] 04.

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4. $\displaystyle\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{\sqrt{y}} 4x^2z^3\,dz\,dy\,dx$

$\displaystyle = \int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{\cancel{4}x^2z^4}{\cancel{4}}\right]_0^{\sqrt{y}} dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1-x} x^2y^2\,dy\,dx$

$\displaystyle = \int_{x=0}^{1}\left[\frac{x^2y^3}{3}\right]_0^{1-x} dx = \frac{1}{3}\int_0^1 x^2(1-x)^3\,dx$

$\displaystyle = \frac{1}{3}\int_0^1 x^2\left(1-3x+3x^2-x^3\right)dx = \frac{1}{3}\int_0^1\left(x^2-3x^3+3x^4-x^5\right)dx$

$\displaystyle = \frac{1}{3}\left[\frac{x^3}{3}-\frac{3x^4}{4}+\frac{3x^5}{5}-\frac{x^6}{6}\right]_0^1 = \frac{1}{3}\left[\frac{1}{3}-\frac{3}{4}+\frac{3}{5}-\frac{1}{6}\right]$

$\displaystyle = \frac{1}{180}$

[9.15] 11.

---

11. $y \in [0, 2]$

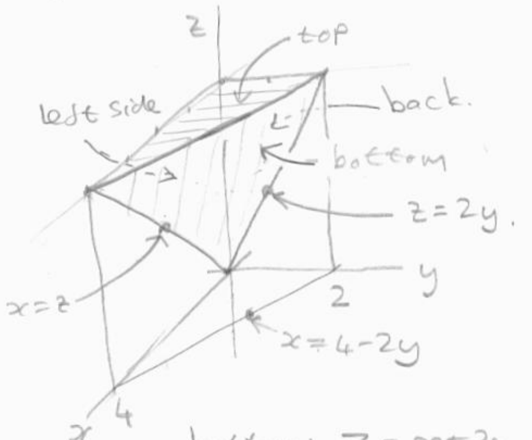
$x \in [0, 4-2y]$

$z \in [x+2y, 4]$ < top $z=4$, bottom $z=x+2y$ or $x+2y-z=0$

$x+2y-z=0$ is a plane going through origin.

At $z=4$: $x+2y=4$: $x=0$ then $y=2$; $y=0$ then $x=4$

$$\int_{y=0}^{2}\int_{x=0}^{4-2y}\int_{z=x+2y}^{4} F \, dz\,dx\,dy$$



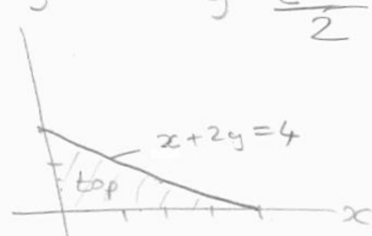
$$\int_{y=0}^{2}\int_{z=2y}^{4}\int_{x=0}^{-2y+z} F \, dx\,dz\,dy$$

bottom: $z=x+2y$

$x=-2y+z$

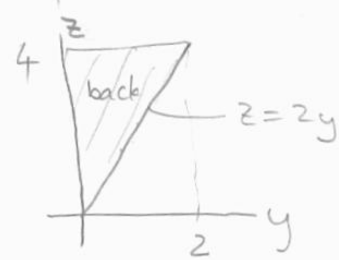
$y=\frac{z-x}{2}$

$$\int_{x=0}^{4}\int_{y=0}^{\frac{4-x}{2}}\int_{z=x+2y}^{4} F \, dz\,dy\,dx$$



$$\int_{z=0}^{4}\int_{x=0}^{z}\int_{y=0}^{\frac{z-x}{2}} F \, dy\,dx\,dz$$

$$\int_{z=0}^{4}\int_{y=0}^{\frac{1}{2}z}\int_{x=0}^{-2y+z} F \, dx\,dy\,dz$$



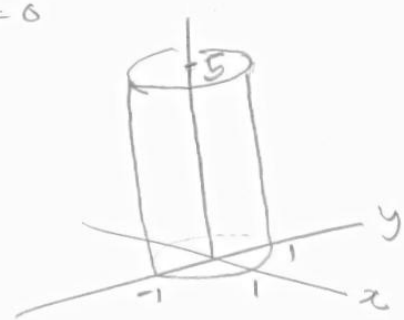
$$\int_{x=0}^{4}\int_{z=x}^{4}\int_{y=0}^{\frac{z-x}{2}} F \, dy\,dz\,dx$$

[9.15] 17.

17.

$$V=\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z=0}^{5} 1\,dz\,dy\,dx$$

Circular cylinder with height 5 and radius 1.



[9.15] 21.

21.

$$\tfrac{1}{2}V=\int_{z=0}^{3}\int_{y=0}^{\sqrt{2}}\int_{x=y^2}^{4-y^2} 1\,dx\,dy\,dz$$

$$=\int_{z=0}^{3}dz\int_{y=0}^{\sqrt{2}}\Big[x\Big]_{y^2}^{4-y^2}dy$$

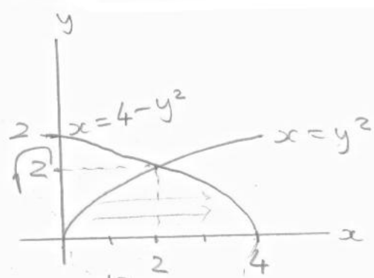
$$=3\cdot\int_0^{\sqrt{2}}\left((4-y^2)-y^2\right)dy$$

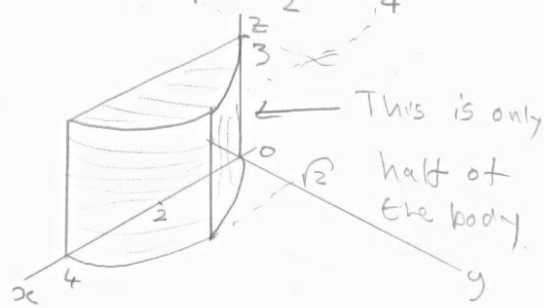
$$=3\int_0^{\sqrt{2}}(4-2y^2)\,dy=3\left[4y-\frac{2y^3}{3}\right]_0^{\sqrt{2}}$$

$$=3\left[4\sqrt{2}-\frac{2\cdot2\sqrt{2}}{3}\right]=\sqrt{2}\left[12-4\right]$$

$$\tfrac{1}{2}V=8\sqrt{2}$$

$$V=16\sqrt{2}$$

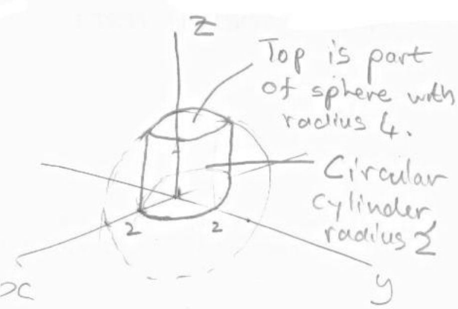




[9.15] 51.

51.

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta$$



$$= 2\pi \int_{r=0}^{2} \Big[r z \Big]_0^{\sqrt{16-r^2}} dr$$

$$x^2+y^2=r^2$$

$$z^2 = 16 - x^2 - y^2$$
$$= 16 - r^2$$

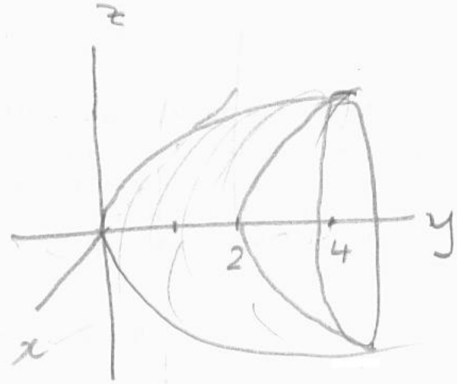
$$= 2\pi \int_{r=0}^{2} \left(r\sqrt{16-r^2} - 0 \right) dr$$

$$= -\pi \int_0^2 -2r\sqrt{16-r^2} \, dr = -\pi \left[\frac{(16-r^2)^{3/2}}{3/2} \right]_0^2$$

$$= -\frac{2\pi}{3} \left[(12)^{3/2} - (16)^{3/2} \right] = -\frac{2\pi}{3} \left[(2\sqrt{3})^3 - 64 \right]$$

$$= \frac{\pi}{3} \left[128 - 16 \cdot 3\sqrt{3} \right] = \frac{(128 - 48\sqrt{3})\pi}{3}$$

[9.15] 54.



54. $y = x^2 + z^2$

$y = \frac{1}{2}x^2 + \frac{1}{2}z^2 + 2$

Intersect at $y = \frac{1}{2}(y) + 2$

$\Leftrightarrow y = 4$

then $x^2 + z^2 = 4$ — circle, radius 2.

$$\left.\begin{array}{l} x = r\cos\theta \\ z = r\sin\theta \\ y = y \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} x^2 + z^2 = r^2 \\ \frac{z}{x} = \tan\theta \\ y = y \end{array}\right.$$

$$\text{Volume} = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{y=r^2}^{\frac{1}{2}r^2+2} 1 \, dy \, r dr d\theta$$

$$= \Big[\theta\Big]_0^{2\pi} \int_{r=0}^{2} \Big[y\Big]_{r^2}^{\frac{1}{2}r^2+2} r dr.$$

$$= 2\pi \int_{r=0}^{2} \left(\tfrac{1}{2}r^2 + 2 - r^2\right) r dr$$

$$= 2\pi \int_0^2 \left(-\tfrac{1}{2}r^3 + 2r\right) dr$$

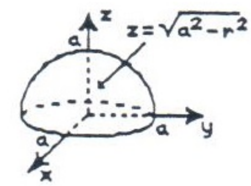
$$= 2\pi \left[-\frac{1}{2}\cdot\frac{r^4}{4} + \frac{2r^2}{2}\right]_0^2$$

$$= 2\pi \left[\frac{-16}{8} + 4\right] = 4\pi$$

[9.15] 55.

55.

The equation is $z = \sqrt{a^2 - r^2}$. By symmetry, $\bar{x} = \bar{y} = 0$.



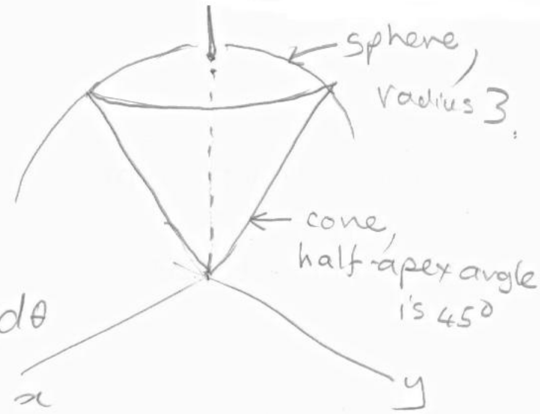
$$m = \int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,\theta = \int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}\,dr\,d\theta$$
$$= \int_0^{2\pi} -\frac{1}{3}(a^2-r^2)^{3/2}\Big|_0^a\,d\theta = \int_0^{2\pi}\frac{1}{3}a^3\,d\theta = \frac{2}{3}\pi a^3$$

$$M_{xy} = \int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^a \frac{1}{2}rz^2\Big|_0^{\sqrt{a^2-r^2}}\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^a r(a^2-r^2)\,dr\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{1}{2}a^2r^2 - \frac{1}{4}r^4\right)\Big|_0^a\,d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1}{4}a^4\,d\theta = \frac{1}{4}\pi a^4$$

$\bar{z} = M_{xy}/m = \dfrac{\pi a^4/4}{2\pi a^3/3} = 3a/8$. The centroid is $(0, 0, 3a/8)$.

[9.15] 75.

75.



$$V = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/4} \int_{r=0}^{3} r^2 \sin\varphi \, dr \, d\varphi \, d\theta$$

$$= \int_{\theta=0}^{2\pi} d\theta \cdot \int_{r=0}^{3} r^2 \, dr \cdot \int_{\varphi=0}^{\pi/4} \sin\varphi \, d\varphi$$

$$= 2\pi \cdot \left[\frac{r^3}{3}\right]_0^3 \cdot \left[-\cos\varphi\right]_0^{\pi/4}$$

$$= 2\pi \cdot \frac{27}{3} \cdot \left[-\frac{1}{\sqrt{2}} - (-1)\right]$$

$$= 2\pi \cdot 9 \left(1 - \frac{1}{\sqrt{2}}\right) = 9\pi\left(2 - \sqrt{2}\right)$$

[9.15] 78.

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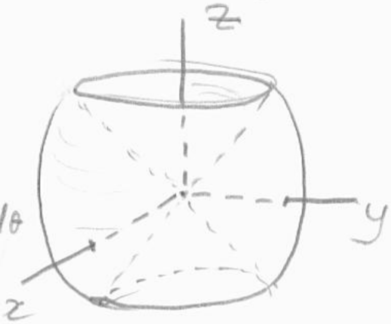
78. $x^2+y^2+z^2=1 \leftarrow$ sphere, radius $=1$

$z^2=x^2+y^2 \longleftarrow$ cone, 45° semi-angle



$$\text{volume} = \int_{\theta=0}^{2\pi}\int_{\varphi=\frac{\pi}{4}}^{3\pi/4}\int_{\rho=0}^{1} 1\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \int_0^{2\pi} d\theta \times \int_{\frac{\pi}{4}}^{3\pi/4} \sin\varphi\,d\varphi \times \int_0^1 \rho^2\,d\rho$$

$$= \Big[\theta\Big]_0^{2\pi}\Big[-\cos\varphi\Big]_{\pi/4}^{3\pi/4}\Big[\frac{\rho^3}{3}\Big]_0^1$$

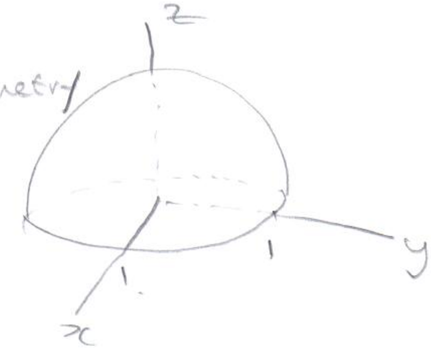
$$= 2\pi \times -\left(-\frac{1}{\sqrt{2}} - \left(+\frac{1}{\sqrt{2}}\right)\right) \times \left[\frac{1}{3}\right]$$

$$= \frac{2\pi}{3}\cdot\frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{3}\pi$$

$$= 2.962$$

[9.15] 80.

[9.15] 80. $\bar{x} = 0, \bar{y} = 0$
because of symmetry



density $= z = \rho \cos\phi$

$$m = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} \int_{\rho=0}^{1} (\rho \cos\varphi)\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_{\theta=0}^{2\pi} d\theta \cdot \int_{\varphi=0}^{\pi/2} \cos\varphi \cdot \sin\varphi \, d\varphi \int_0^1 \rho^3 d\rho.$$

$$= 2\pi \left[\frac{(\sin\varphi)^2}{2}\right]_0^{\pi/2} \left[\frac{\rho^4}{4}\right]_0^1 = 2\pi \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{\pi}{4}$$

$$\bar{z}\, m = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} \int_{\rho=0}^{1} (\rho \cos\varphi)^2 \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos^2\varphi \sin\varphi \, d\varphi \cdot \int_0^1 \rho^4 d\rho.$$

$$= 2\pi \left[-\frac{(\cos\varphi)^3}{3}\right]_0^{\pi/2} \left[\frac{\rho^5}{5}\right]_0^1 = 2\pi \left[\frac{1}{3}\right]\left(\frac{1}{5}\right)$$

$$= \frac{2\pi}{15}$$

$$\bar{z} = \frac{2\pi/15}{\pi/4} = \frac{8}{15} = 0.533$$