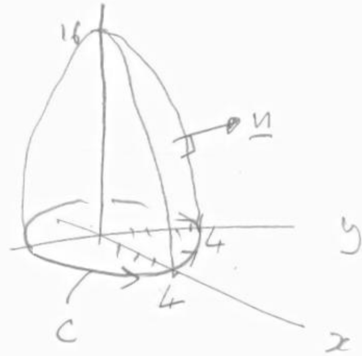
TUT 10: [9.14] 2, 4, 6, 11, 13, 16, 18.

[9.14] 02a.

2.



$$\oint_C \underline{F}\cdot d\underline{r}$$

$$= \oint_C (2z\,dx - 3x\,dy + 4y\,dz)$$

Parameterise: $z = 0$ $\quad dz = 0$

$x = 4\cos t$ $\quad dx = -4\sin t$

$y = 4\sin t$ $\quad dy = 4\cos t$.

$$= \int_{t=0}^{2\pi} 0 - 3(4\cos t)(4\cos t) + 0$$

$$= -48\int_0^{2\pi} \cos^2 t\,dt = -48\left[\int_0^{2\pi} \frac{1}{2}\,dt + \int_0^{2\pi} \frac{1}{2}\cos(2t)\,dt\right]$$

$$= -48\cdot\frac{1}{2}\left[2\pi\right] = -48\pi$$

[9.14] 02b.

2 continued:

$$\underline{\nabla} \times \underline{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} 2z \\ -3x \\ 4y \end{bmatrix} = \begin{bmatrix} 4-0 \\ 2-0 \\ -3-0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$$

$$g = z + x^2 + y^2 - 16$$

$$\underline{n}\, dS = \underline{\nabla} g \, dA = (2x\,\underline{i} + 2y\,\underline{j} + \underline{k})\, dA.$$

$$(\underline{\nabla} \times \underline{F}) \cdot \underline{n}\, dS = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix} = 8x + 4y - 3$$

$$\iint_S (8x + 4y - 3)\, dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{4} (8r\cos\theta + 4r\sin\theta - 3)\, r\, dr\, d\theta.$$

$$= \int_0^{2\pi} (8\cos\theta + 4\sin\theta)\, d\theta \cdot \int_{r=0}^{4} r^2\, dr \;-\; 3 \int_0^{2\pi} d\theta \cdot \int_0^4 r\, dr$$

$$= \quad 0 \quad - 3 \cdot (2\pi) \cdot \left(\frac{4^2}{2} \right) = -48\pi$$

[9.14] 04.

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4. $\oint_C x\,dx + y\,dy + z\,dz$

Parameterize: $x = \cos t$, $dx = \sin t\,dt$
$y = \sin t$, $dy = \cos t\,dt$
$z = 0$, $dz = 0$

$$\int_{t=0}^{2\pi} \cos t \cdot \sin t\,dt + \sin t(-\cos t)\,dt + 0$$

$$= \int_{t=0}^{2\pi} 0\,dt = 0$$

$$\underline{\nabla} \times \underline{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 0-0 \\ 0-0 \\ 0-0 \end{pmatrix} = \underline{0}$$

therefore

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{n}\,dS = 0$$

[9.14] 06.

6. $$\underline{\nabla} \times \underline{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} z^2 y \cos xy \\ z^2 x(1+\cos xy) \\ 2z \sin xy \end{bmatrix}$$

$$= \begin{bmatrix} 2zx\cos xy - 2zx(1+\cos xy) \\ 2zy\cos xy - 2zy\cos xy \\ z^2(1+\cancel{\cos xy}) + \cancel{z^2xy(-\sin xy)} - \cancel{z^2\cos xy} - \cancel{z^2yx \times (-\sin xy)} \end{bmatrix}$$

$$= \begin{bmatrix} -2zx \\ 0 \\ z^2 \end{bmatrix}$$

$$\underline{n}\,dS = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} dA \qquad (\underline{\nabla} \times \underline{F}) \cdot \underline{n}\,dS = z^2\,dA$$

$$\oint_C \underline{F} \cdot d\underline{r} = \iint_R z^2\,dA = \int_{x=0}^{2} \int_{y=0}^{1} (1-y)^2\,dy\,dx$$

$$= \int_0^2 dx \cdot \int_0^1 (1-y)^2\,dy = 2 \cdot \left[-\frac{(1-y)^3}{3} \right]_0^1$$

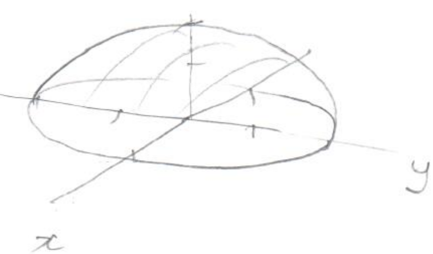
$$= -\frac{2}{3}\left[0 - 1\right] = \frac{2}{3}$$

[9.14] 11a.

[9.14 – 5]

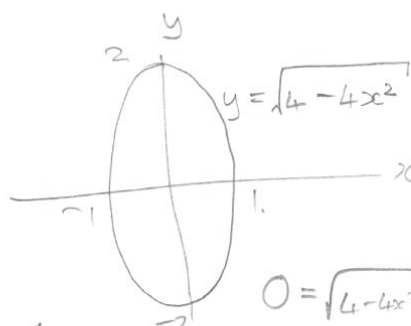
11. $$(\underline{\nabla} \times \underline{F}) = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} x \\ x^3y^2 \\ z \end{bmatrix} = \begin{bmatrix} 0-0 \\ 0-0 \\ 3x^2y^2 - 0 \end{bmatrix}$$

$$g = z - \sqrt{4 - 4x^2 - y^2}$$



$$\underline{\nabla} g = \frac{+8x}{2\sqrt{4-4x^2-y^2}}\underline{i} + \frac{2y}{2\sqrt{4-4x^2-y^2}}\underline{j} + \underline{k}$$

$$\underline{n}\, dS = \begin{bmatrix} 4x/\Gamma \\ y/\Gamma \\ 1 \end{bmatrix} dA$$



$$0 = \sqrt{4-4x^2-y^2}$$
$$4x^2 + y^2 = 4$$
$$y^2 = 4(1-x^2)$$

$$(\underline{\nabla} \times \underline{F}) \cdot \underline{n}\, dS = 3x^2y^2\, dA$$

$3x^2y^2$ is quadrilaterally symmetric around the origin, so we can us symmetry.

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{n}\, dS = 4 \iint_{\frac{1}{4}R} 3x^2y^2\, dA$$

$$= 12 \int_{x=0}^{1} \int_{y=0}^{2\sqrt{1-x^2}} x^2y^2\, dy\, dx$$

[9.14] 11b.

11 Continued:

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{n}\, dS = 12 \int_{x=0}^{1} \left[\frac{x^2 y^3}{3} \right]_0^{2\sqrt{1-x^2}} dx$$

$$= 4 \int_0^1 x^2 \left(8(1-x^2)^{3/2} - 0 \right) dx$$

$$= 32 \int_0^1 x^2 (1-x^2)^{3/2}\, dx$$

Let $x = \sin t$, $dx = \cos t\, dt$

$x \in [0,1] \Rightarrow t \in [0, \pi/2]$

$$= 32 \int_0^{\pi/2} \sin^2 t \left(\sqrt{\cos t}\right)^{3/2} \cos t\, dt$$

$$= 32 \underbrace{\int_0^{\pi/2} \sin^2 t \cos^4 t\, dt}$$

Difficult integral, but here is a possible way of doing it by hand.

$$A = \sin^2 t \cos^4 t = \frac{1}{4}(2 \sin t \cos t)^2 \cos^2 t$$

$$= \frac{1}{4} (\sin 2t)^2 \left[\frac{1}{2} + \frac{1}{2} \cos 2t \right]$$

$$= \frac{1}{8} \left[(\sin 2t)^2 + \frac{1}{2} (\sin 2t)^2 (2 \cos 2t) \right]$$

[9.14] 11c.

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11 Continued further

$$A = \frac{1}{8}\left[\frac{1}{2} - \frac{1}{2}\cos(4t) + \frac{1}{2}(\sin 2t)^2\underbrace{(2\cos 2t)}_{\text{derivative of}}\right]$$

$$A = \frac{1}{16}\left[1 - \cos(4t) + (\sin 2t)^2(2\cos 2t)\right]$$

$$\iint_S (\nabla \times F)\cdot n\, dS = \frac{32}{16}\left[\int_0^{\pi/2} 1\, dt - \int_0^{\pi/2}\cos(4t)\,dt + \int_0^{\pi/2}(\sin 2t)^2(2\cos 2t)\,dt\right]$$

$$= 2\left[\frac{\pi}{2} - \left[\frac{\sin(4t)}{4}\right]_0^{\pi/2} + \left[\frac{(\sin(2t))^3}{3\cdot 2}\right]_0^{\pi/2}\right]$$

$$= \pi - 0 + 0$$

$$= \pi$$

[9.14] 13.

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13. $\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{n}\, dS = \oint_C \underline{F} \cdot d\underline{r}$ $\qquad z = \frac{x^2}{4} + y^2$

$$= \oint_C 6yz\,dx + 5x\,dy + yze^{x^2}\,dz$$

Parameterize: $z = 4$

$$x = 4\cos t$$

$$4 = \frac{(4\cos t)^2}{4} + y^2$$

$$\Rightarrow \quad y^2 = 4 - 4\cos^2 t$$
$$= 4(1 - \cos^2 t)$$
$$= 4\sin^2 t$$
$$y = 2\sin t$$

$z = 4$

$$4 = \frac{x^2}{4} + y^2$$

$$1 = \frac{x^2}{16} + \frac{y^2}{4} = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2$$

$$dx = -4\sin t\,dt$$
$$dy = 2\cos t\,dt$$
$$dz = 0$$

$$\oint \underline{F} \cdot d\underline{r} = \int_{t=0}^{2\pi} 6 \cdot (2\sin t)(4)(-4\sin t)\,dt$$

$$+ 5 \cdot (4\cos t)(2\cos t)\,dt + ? \cdot 0$$

$$= \int_{t=0}^{2\pi} \left(-192\sin^2 t + 40\cos^2 t\right) dt$$

$$= \int_0^{2\pi} \left(-232\sin^2 t + 40(\sin^2 t + \cos^2 t)\right) dt \quad = -152\pi$$

$$= \int_0^{2\pi} 40\,dt - 232 \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right) dt = \left(40 - \frac{232}{2}\right) \int_0^{2\pi} 1\,dt$$

[9.14] 16a.

16. $C: \left.\begin{aligned} x &= \cos t \\ y &= \sin t \\ z &= \sin t \end{aligned}\right\} \quad t \in [0, 2\pi]$

$$\iint_S (\nabla \times \underline{F}) \cdot \underline{n}\, dS = \oint_C \underline{F} \cdot d\underline{r}$$

$$= \oint (2xy^2z)\,dx + (2x^2yz)\,dy + (x^2y^2 - 6x)\,dz$$

$$= \int_0^{2\pi} (2\cos t \cdot \sin^2 t \cdot \sin t)(-\sin t\, dt)$$

$$+ (2\cos^2 t \cdot \sin t \cdot \sin t)(\cos t\, dt)$$

$$+ (\cos^2 t \cdot \sin^2 t - 6\cos t)(\cos t\, dt)$$

$$= \int_0^{2\pi} \Big(-2\sin^4 t \cos t + 2\cos^3 t \cdot \sin^2 t + \cos^3 t \cdot \sin^2 t$$
$$- 6\cos^2 t\Big)\,dt$$

$$= \int_0^{2\pi} \Big[-2\sin^4 t \cdot \cos t + 3(1 - \sin^2 t) \cdot \cos t \cdot \sin^2 t$$
$$- 6\cos^2 t\Big]\,dt$$

[9.14] 16b.

16 continued

$$= \int_0^{2\pi} \left(-5 \sin^4 t \cdot \cos t - 3 \sin^2 t \cos t\right) dt$$

$$-6 \int_0^{2\pi} \cos^2 t \, dt$$

$$= -5 \frac{(\sin t)^5}{5} \Big|_0^{2\pi} - 3 \frac{(\sin t)^3}{3} \Big|_0^{2\pi}$$

$$-6 \int_0^{2\pi} \frac{1}{2}\left(\cos(2t) + 1\right) dt$$

$$= 0 + 0 - \frac{6}{2}\left[\frac{\sin(2t)}{2} + t\right]_0^{2\pi}$$

$$= -3\left[2\pi - 0\right] = -6\pi$$

[9.14] 18a.

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18. $\underline{F} = xyz\,\underline{k}$

$$\underline{\nabla}\times\underline{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ xyz \end{bmatrix} = \begin{bmatrix} xz-0 \\ 0-yz \\ 0 \end{bmatrix}.$$

(a) $z = 1-x^2-y^2$ $\quad dS = \sqrt{1+4x^2+4y^2}\,dA$

$$\underline{n} = \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix} \frac{1}{\sqrt{1+4x^2+4y^2}}$$

$$\iint_S (\underline{\nabla}\times\underline{F})\cdot\underline{n}\,dS = \iint_R \big(2x(xz)+2y(-yz)\big)\,dA$$

$$= \iint_R 2x^2(1-x^2-y^2) - 2y^2(1-x^2-y^2)\,dA$$

$$= \iint_R \left(2x^2-2x^4-\cancel{2x^2y^2}-2y^2+\cancel{2x^2y^2}+2y^4\right)dA$$

$$= 2\iint_R \left(x^2-y^2-x^4+y^4\right)dA$$

$$= 2\int_{r=0}^{1}\int_{\theta=0}^{2\pi} \left(r^2\cos^2\theta - r^2\sin^2\theta - r^4\cos^4\theta + r^4\sin^4\theta\right) r\,dr\,d\theta$$

$$= 2\int_0^1\int_{\theta=0}^{2\pi} r^3(\cos^2\theta-\sin^2\theta) + r^5\underbrace{(\sin^4\theta-\cos^4\theta)}_{(\sin^2\theta+\cos^2\theta)(\sin^2\theta-\cos^2\theta)}\Big)\,dr\,d\theta$$

[9.14] 18b.

18 vervolg:

$$= \int_0^1 (2r^3 - 2r^5)\,dr \cdot \underbrace{\int_0^{2\pi} \cos(2\theta)\,d\theta}_{=0} = 0$$

(b) Eenvoudiger: $z = 0$, Sirkel met radius 1:

$dS = dA$

$\underline{n} = \underline{k}$

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{n}\,dS = \iint_R \begin{bmatrix} xz \\ -yz \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dA$$

$$= \iint_R 0\,dA = 0$$

(c) $\left.\begin{aligned} x &= \cos t \\ y &= \sin t \\ z &= 0 \end{aligned}\right\} \quad t \in [0, 2\pi]$

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{n}\,dS = \oint_C \underline{F} \cdot d\underline{r} = \oint_C (xyz)\,dz = \oint_C 0\,dz = 0$$