TUT 9: [9.13] 4, 7, 10, 17, 21, 25, 35, 36.

[9.13] 04.

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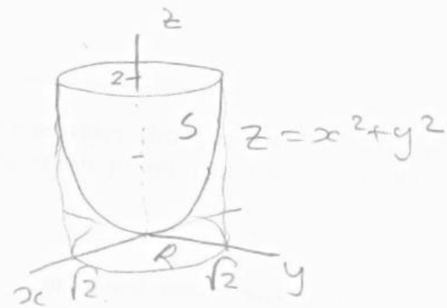
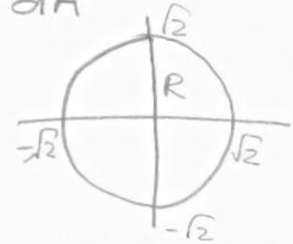
4. $f(x,y) = x^2+y^2$

$$dS = \sqrt{1+(2x)^2+(2y)^2}\,dA$$
$$= \sqrt{1+4x^2+4y^2}\,dA$$



$$\text{area} = \iint_S 1\,dS = \iint_R \sqrt{1+4x^2+4y^2}\,dA$$



$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{2}} \sqrt{1+4r^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} 1\,d\theta \times \int_0^{\sqrt{2}} r\sqrt{1+4r^2}\,dr = 2\pi \cdot \frac{1}{8}\int_0^{\sqrt{2}} 8r\sqrt{1+4r^2}\,dr$$

$$= \frac{2\pi}{8}\cdot\left[\frac{(1+4r^2)^{\frac{3}{2}}}{3/2}\right]_0^{\sqrt{2}} = \frac{\pi}{\cancel{4}_2}\cdot\frac{\cancel{2}}{3}\cdot\left[(9)^{3/2} - (1)^{3/2}\right]$$

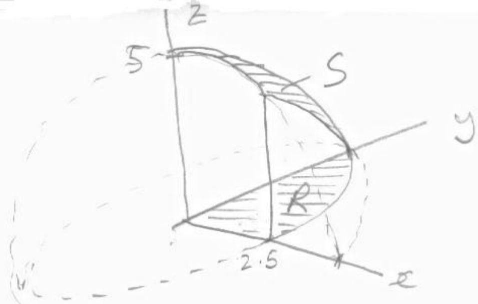
$$= \frac{\pi}{6}\left[27-1\right] = \frac{26\pi}{6} = \frac{13\pi}{3}$$

[9.13] 07.

7. $4x^2+y^2=25$

$$\frac{x^2}{25/4}+\frac{y^2}{25}=1$$

$$\frac{x^2}{(5/2)^2}+\frac{y^2}{5^2}=1$$

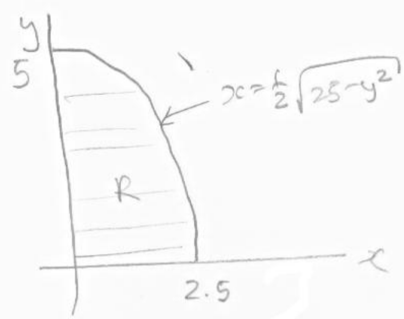


$$\text{Area}=\iint_R \frac{5}{\sqrt{25-x^2-y^2}}\,dA$$

$$z=f(x,y)=\sqrt{25-x^2-y^2}$$

$$\frac{\partial f}{\partial x}=\frac{-x}{\sqrt{\quad}},\quad \frac{\partial f}{\partial y}=\frac{-y}{\sqrt{\quad}}$$

$$dS=\frac{5}{\sqrt{25-x^2-y^2}}\,dA$$



$$\text{Area}=\int_{y=0}^{5}\int_{x=0}^{\frac{1}{2}\sqrt{25-y^2}}\frac{5}{\sqrt{25-x^2-y^2}}\,dx\,dy$$

$$=\int_{y=0}^{5} 5\left[\arcsin\left(\frac{x}{\sqrt{25-y^2}}\right)\right]_0^{\frac{1}{2}\sqrt{25-y^2}} dy$$

$$=5\int_0^5\left[\arcsin\left(\tfrac{1}{2}\right)-\arcsin(0)\right]dy$$

$$=5\arcsin\left(\tfrac{1}{4}\right)\cdot\left[y\right]_0^5 = 25\arcsin\left(\tfrac{1}{2}\right)=25\cdot\frac{\pi}{6}$$

$$=13.09.$$

[9.13] 10.

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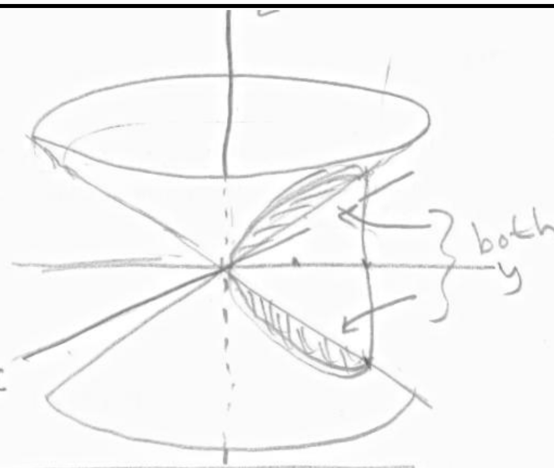
10. $z^2 = \frac{1}{4}(x^2+y^2)$

$(x-1)^2 + y^2 = 1$



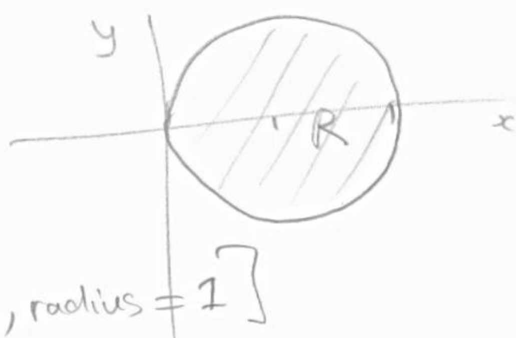
$f(x,y) = \frac{1}{2}\sqrt{x^2+y^2}$

$$dS = \sqrt{1 + \left(\frac{\frac{1}{2}(\cancel{2}x)}{\cancel{2}\sqrt{x^2+y^2}}\right)^2 + \left(\frac{\frac{1}{2}(\cancel{2}y)}{\cancel{2}\sqrt{x^2+y^2}}\right)^2}\, dA$$

$$= \sqrt{1 + \frac{\frac{1}{4}x^2}{x^2+y^2} + \frac{\frac{1}{4}y^2}{x^2+y^2}}\, dA$$

$$= \sqrt{\frac{x^2+y^2+\frac{1}{4}x^2+\frac{1}{4}y^2}{x^2+y^2}}\, dA$$

$$= \sqrt{\frac{\frac{5}{4}(x^2+y^2)}{(x^2+y^2)}}\, dA = \frac{\sqrt{5}}{2}\, dA$$



$$\text{area} = 2\iint_R 1\, \frac{\sqrt{5}}{2}\, dA$$

$$= \frac{2}{2}\sqrt{5}\left[\text{area of disc, radius} = 1\right]$$

$$= \sqrt{5}\,(\pi 1^2) = \sqrt{5}\,\pi$$

[9.13] 17.

17. $G = xz^3$, $z = f(x,y) = \sqrt{x^2+y^2}$

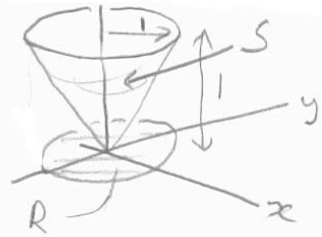
$$\iint G\,dS = \iint_S xz^3\,dS$$

$$= \iint_R x\,(x^2+y^2)^{3/2}\sqrt{2}\,dA$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\cos\theta \cdot r^3\sqrt{2}\; r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\cos\theta\,d\theta \times \int_0^1 r^5\,dr$$

$$= \sqrt{2} \times 0 \times ? = 0$$



$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{x^2+y^2}},\quad \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{x^2+y^2}}$$

$$dS = \sqrt{1 + \frac{x^2}{r^2} + \frac{y^2}{r^2}}\,dA$$

$$= \sqrt{\frac{x^2+y^2+x^2+y^2}{x^2+y^2}}\,dA$$

$$= \sqrt{2}\,dA$$

[9.13] 21.

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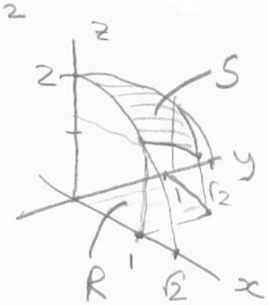
21. $G = xy$, $z = f(x,y) = 2 - \frac{1}{2}x^2 - \frac{1}{2}y^2$



$$\iint_S xy\,dS = \iint_R xy\sqrt{1+x^2+y^2}\,dA$$

$$= \int_{y=0}^{1}\int_{x=0}^{1} xy\sqrt{1+x^2+y^2}\,dx\,dy$$

$$= \int_{y=0}^{1} \frac{1}{2}y\left[\frac{(1+x^2+y^2)^{3/2}}{3/2}\right]_0^1 dy$$

$$\frac{\partial f}{\partial x} = -x, \quad \frac{\partial f}{\partial y} = -y$$

$$dS = \sqrt{1+x^2+y^2}\,dA$$

$$= \frac{1}{3}\int_0^1 y\left[(1+1+y^2)^{3/2} - (1+0+y^2)^{3/2}\right]dy$$

$$= \frac{1}{3}\cdot\frac{1}{2}\left[\frac{(2+y^2)^{5/2}}{5/2} - \frac{(1+y^2)^{5/2}}{5/2}\right]_0^1 = \frac{1}{15}\left(3^{5/2} - 2^{5/2} - 2^{5/2} + 1\right)$$

$$= \frac{1}{15}\left(3^{5/2} + 1 - 2^{7/2}\right) = 0.352$$

[9.13] 25.

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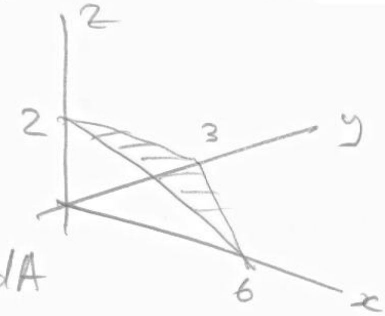
25. $x+2y+3z=6$, $y=f(x,z)$



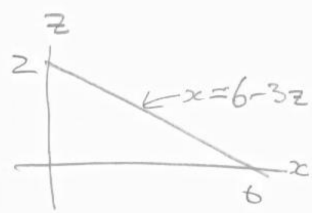
$y = 3 - \frac{1}{2}x - \frac{3}{2}z$

$dS = \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$

$= \sqrt{1+\left(-\frac{1}{2}\right)^2 + \left(-\frac{3}{2}\right)^2}\, dA = \sqrt{\frac{7}{2}}\, dA$

$\iint_S (3z^2 + 4yz)\, dS = \iint_R \left(3z^2 + 4z\left(3 - \frac{1}{2}x - \frac{3}{2}z\right)\right)\sqrt{\frac{7}{2}}\, dA$

$= \sqrt{\frac{7}{2}} \iint_R (12z - 2zx - 3z^2)\, dx\, dz$



$= \sqrt{\frac{7}{2}} \int_{z=0}^{2} \int_{x=0}^{6-3z} (12z - 2zx - 3z^2)\, dx\, dz$

$= \sqrt{\frac{7}{2}} \int_{z=0}^{2} \left[(12z - 3z^2)x - \frac{2zx^2}{2}\right]_0^{6-3z} dz$

$= \sqrt{\frac{7}{2}} \int_0^2 \left[(12z-3z^2)(6-3z) - z(6-3z)^2\right] dz$

$= \sqrt{\frac{7}{2}} \int_0^2 (36z - 18z^2)\, dz = \sqrt{\frac{7}{2}} \left[\frac{36z^2}{2} - \frac{18z^3}{3}\right]_0^2$

$= \sqrt{\frac{7}{2}} \left[18(4) - 6(8)\right] = 24\sqrt{\frac{7}{2}} = 24\sqrt{\frac{2 \cdot 7}{4}}$

$= \frac{24}{2}\sqrt{14} = 12\sqrt{14}$

[9.13] 35.

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35. $\text{flux} = \iint_S \underline{F} \cdot \underline{n}\, dS$

$= \iint_{S_1} \underline{F} \cdot \underline{n}\, dS + \iint_{S_2} \underline{F} \cdot \underline{n}\, dS$

$= \iint_{S_1} \begin{bmatrix} y^2 \\ x^2 \\ 5z \end{bmatrix} \cdot \underline{k}\, dA \quad +$

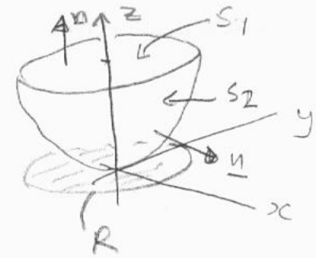
$\iint_{S_2} \begin{bmatrix} y^2 \\ x^2 \\ 5z \end{bmatrix} \cdot \begin{bmatrix} 2x \\ 2y \\ -1 \end{bmatrix} dA.$

$= \iint_R \underset{z=1}{\underset{\uparrow}{5z}}\, dA + \iint_R (2xy^2 + 2yx^2 \underset{z=x^2+y^2}{\underset{\uparrow}{-5z}})\, dA$

for S_1:

$dS = dA$

$\underline{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{k}$



for S_2: $g = z - x^2 - y^2$

$\underline{\nabla} g = -2x\underline{i} - 2y\underline{j} + \underline{k}$

$\underline{n} = \frac{1}{\sqrt{1+4x^2+4y^2}} \begin{bmatrix} -2x \\ -2y \\ 1 \end{bmatrix} \leftarrow$ but this is inward

$\underline{n} = \frac{1}{\sqrt{1+4x^2+4y^2}} \begin{bmatrix} 2x \\ 2y \\ -1 \end{bmatrix} \leftarrow$ outward normal

$dS = \sqrt{1+4x^2+4y^2}\, dA.$

$\text{flux} = \iint_R 5\, dA + \iint_R \left(2xy^2 + 2yx^2 - 5(x^2+y^2)\right) dA$

$= 5\,(\text{area of circle}) + \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \Big(2r^3 \cos\theta \sin^2\theta$

$+ 2r^3 \sin\theta \cos^2\theta - 5r^2 \Big)\, r\, dr\, d\theta$

$= 5\pi + \int_{\theta=0}^{2\pi} \left(2\cos\theta \sin^2\theta + 2\sin\theta\cos^2\theta\right) d\theta \int_{r=0}^{1} r^4\, dr$

$- 5 \int_{\theta=0}^{2\pi} 1\, d\theta \int_{r=0}^{1} r^3\, dr$

$= 5\pi + \frac{2}{3}\left[\sin^3\theta - \cos^3\theta\right]_0^{2\pi} \cdot \frac{1}{5} - 5\,(2\pi)\,\frac{1}{4}$

$= 5\pi + 0 - \frac{5\pi}{2} = \frac{5\pi}{2}$

[9.13] 36.

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36. For S_1: $g(x,y,z) = x^2 + y^2 + z - 4$, $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$, $|\nabla g| = \sqrt{4x^2 + 4y^2 + 1}$;

$\mathbf{n}_1 = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}$; $\mathbf{F} \cdot \mathbf{n}_1 = 6z^2/\sqrt{4x^2 + 4y^2 + 1}$; $z_x = -2x$, $z_y = -2y$,

$dS_1 = \sqrt{1 + 4x^2 + 4y^2}\, dA$. For S_2: $g(x,y,z) = x^2 + y^2 - z$, $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$,

$|\nabla g| = \sqrt{4x^2 + 4y^2 + 1}$; $\mathbf{n}_2 = \dfrac{2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}}{\sqrt{4x^2 + y^2 + 1}}$; $\mathbf{F} \cdot \mathbf{n}_2 = -6z^2/\sqrt{4x^2 + 4y^2 + 1}$; $z_x = 2x$, $z_y = 2y$,

$dS_2 = \sqrt{1 + 4x^2 + 4y^2}\, dA$. Using polar coordinates and R: $x^2 + y^2 \le 2$ we have

$$\text{Flux} = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS_1 + \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_2\, dS_2 = \iint_R 6z^2\, dA + \iint -6z^2\, dA$$

$$= \iint_R [6(4 - x^2 - y^2)^2 - 6(x^2 + y^2)^2]\, dA = 6 \int_0^{2\pi} \int_0^{\sqrt{2}} [(4 - r^2)^2 - r^4]\, r\, dr\, d\theta$$

$$= 6 \int_0^{2\pi} \left[-\frac{1}{6}(4 - r^2)^3 - \frac{1}{6} r^6 \right] \Big|_0^{\sqrt{2}} d\theta = -\int_0^{2\pi} [(2^3 - 4^3) + (\sqrt{2})^6]\, d\theta = \int_0^{2\pi} 48\, d\theta = 96\pi.$$