TUT 5: [9.8] 3, 6, 9, 16, 19, 25, 27, 31, 34. [9.9] 3, 5, 13, 14, 17, 18, 22, 25.

[9.8] 03.

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3. $C: \left.\begin{array}{l} x = t \\ y = 2t+1 \end{array}\right\} \; t \in [-1, 0]$, $G(x,y) = 3x^2 + 6y^2$

$G(x(t), y(t)) = 3t^2 + 6(2t+1)^2$
$= 3t^2 + 24t^2 + 24t + 6$
$= 27t^2 + 24t + 6$

$\int_C G\,dx$

$= \int_{-1}^{0} (27t^2 + 24t + 6)\,dt$

$= 27 \frac{t^3}{3}\Big|_{-1}^{0} + 24 \frac{t^2}{2}\Big|_{-1}^{0} + 6t\Big|_{-1}^{0} = +\frac{27}{3} - \frac{24}{2} + 6$

$= 3$

$\int_C G\,dy = \int_{-1}^{0} (27t^2 + 24t + 6)\,2dt = 2\int_C G\,dx = 6$

$ds = \sqrt{1 + 2^2}\,dt = \sqrt{5}\,dt$

$\int_C G\,ds = \sqrt{5}\int_{-1}^{0} (27t^2 + 24t + 6)\,dt = 3\sqrt{5}$

[9.8] 06.

6. $\displaystyle\int_C 4xyz\,dx = \int_0^1 4\left(\tfrac{1}{3}t^3\right)t^2(2t)(t^2\,dt)$

$\displaystyle = \frac{8}{3}\int_0^1 t^8\,dt = \frac{8}{3}\left[\frac{t^9}{9}\right]_0^1 = \frac{8}{27}$

$\displaystyle \frac{dx}{dt} = 3\frac{t^2}{3} = t^2$

$\displaystyle \frac{dy}{dt} = 2t$

$\displaystyle \frac{dz}{dt} = 2$

$\displaystyle \frac{ds}{dt} = \sqrt{(t^2)^2 + (2t)^2 + 2^2}$

$\displaystyle ds = \sqrt{t^4 + 4t^2 + 4}$

$\displaystyle\int_C 4xyz\,dy = \int_0^1 \frac{8}{3}t^6(2t)\,dt$

$\displaystyle = \frac{8\times 2}{3}\left[\frac{t^8}{8}\right]_0^1 = \frac{2}{3}$

$\displaystyle\int_C 4xyz\,dz = \int_0^1 \frac{8}{3}t^6\, 2\,dt = \frac{16}{3}\left[\frac{t^7}{7}\right]_0^1 = \frac{16}{21}$

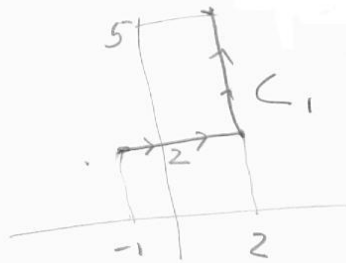
$\displaystyle\int_C 4xyz\,ds = \int_0^1 \frac{8}{3}t^6\sqrt{(t^2+2)^2}\,dt$

$\displaystyle = \int_0^1 \frac{8}{3}t^6(t^2+2)\,dt = \frac{8}{3}\left[\frac{t^9}{9} + \frac{2t^7}{7}\right]_0^1$

$\displaystyle = \frac{8}{3}\left[\frac{1}{9} + \frac{2}{7}\right] = \frac{200}{189}$

[9.8] 09.

9. $A = \int_{C_1} (2x+y)\,dx + xy\,dy$



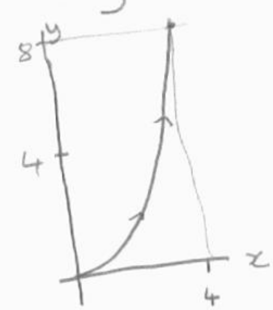
$$A = \int_{x=-1}^{2} (2x+2)\,dx + 0 + 0 + \int_{y=2}^{5} 2y\,dy$$

$$= \left[\frac{2x^2}{2} + 2x\right]_{-1}^{2} + \left[\frac{2y^2}{2}\right]_{2}^{5} = 4+4-1+2+25-4$$

$$= 30$$

[9.8] 16.

16. $\int_C -y^2\,dx + xy\,dy$ $\quad C: \left.\begin{array}{l} x = 2t \\ y = t^3 \end{array}\right\} \ t \in [0,2]$



$dx = 2dt$

$dy = 3t^2dt$

$$= \int_0^2 -(t^3)^2\, 2dt + (2t)(t^3)\, 3t^2dt$$

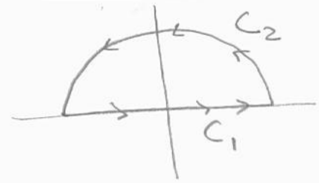
$$= \int_0^2 (-2t^6 + 6t^6)\,dt$$

$$= 4\int_0^2 t^6\,dt = 4\left[\frac{t^7}{7}\right]_0^2$$

$$= 4 \times \frac{128}{7} = \frac{512}{7} = 73.143$$

[9.8] 19.

19. $\oint_C (x^2+y^2)\,dx - 2xy\,dy$



C_1: $\left.\begin{array}{l} x = t \\ y = 0 \end{array}\right\}$ $t \in [-2,2]$

C_2: $\left.\begin{array}{l} x = 2\cos t \\ y = 2\sin t \end{array}\right\}$ $t \in [0,\pi]$

$$= \int_{-2}^{2} (t^2+0)\,dt - 2t(0)$$

$$+ \int_0^{\pi} \left[(2\cos t)^2 + (2\sin t)^2\right] 2(-\sin t)\,dt$$

$$- 2(2\cos t)(2\sin t)(2\cos t)\,dt$$

$$= \int_{-2}^{2} t^2\,dt + \int_0^{\pi} (-8\sin t - 16\cos^2 t \sin t)\,dt$$

$$= \left.\frac{t^3}{3}\right|_{-2}^{2} + \left.8\cos t\right|_0^{\pi} + \left.16\,\frac{\cos^3 t}{3}\right|_0^{\pi} = \frac{16}{3} + 8(-2) + \frac{16}{3}(-2)$$

$$= -\frac{64}{3}$$

[9.8] 25.

25. C_1: $\left.\begin{array}{l} x=2t \\ y=3t \\ z=4t \end{array}\right\} t\in[0,1]$

C_2: $\left.\begin{array}{l} x=2+4t \\ y=3+5t \\ z=4+t \end{array}\right\} t=[0,1]$

$$\int_{C_1} y\,dx+z\,dy+x\,dz + \int_{C_2} y\,dx+z\,dy+x\,dz$$

$$= \int_{t=0}^{1} 3t\,2dt + 4t\,3dt + 2t\,4dt$$

$$+ \int_{t=0}^{1} (3+5t)4dt + (4+t)5dt + (2+4t)dt$$

$$= \int_{t=0}^{1} (6t + 12t + 8t + 12 + 20t + 20 + 5t + 2 + 4t)dt$$

$$= \int_{t=0}^{1} (55t + 34)dt = \frac{55t^2}{2} + 34t \Big|_0^1$$

$$= \frac{55}{2} - 0 + 34 - 0$$

$$= \frac{123}{2}$$

ABC trial 4

[9.8] 27.

27. $\int_C y\,dx + z\,dy + x\,dz$

$C_1: \left.\begin{array}{l} x=t \\ y=0 \\ z=0 \end{array}\right\} t\in[0,6]$

$C_2: \left.\begin{array}{l} x=6 \\ y=0 \\ z=t \end{array}\right\} t\in[0,5]$

$C_3: \left.\begin{array}{l} x=6 \\ y=t \\ z=5 \end{array}\right\} t\in[0,8]$

$$= \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$= \int_{t=0}^{6} 0\,dt + 0\,dt + t\cdot 0\,dt$$

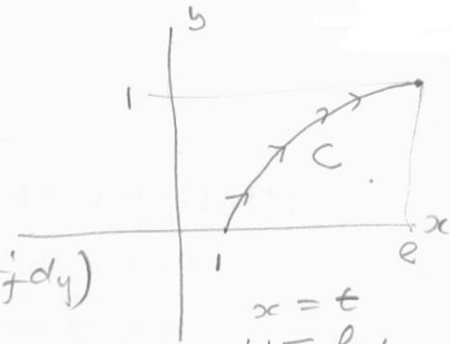
$$+ \int_{t=0}^{5} 0\cdot 0\,dt + t\cdot 0\,dt + 6\cdot 1\,dt$$

$$+ \int_{t=0}^{8} t\cdot 0\,dt + 5\cdot 1\,dt + 6\cdot 0\,dt$$

$$= 0 + 6\big[t\big]_0^5 + 5\big[t\big]_0^8 = 30+40 = 70$$

[9.8] 31.

31.
$$W = \int_C \underline{F} \cdot d\underline{r}$$



$$= \int_C (y\underline{\hat{i}} + x\underline{\hat{j}}) \cdot (\underline{\hat{i}}\,dx + \underline{\hat{j}}\,dy)$$

$x = t$

$y = \ln t$

$t \in [1, e]$

$$= \int_C y\,dx + x\,dy$$

$\frac{dy}{dt} = \frac{1}{t}$

$$= \int_{t=1}^{e} \ln t \; dt + t \cdot \frac{1}{t}\,dt$$

$$= \int_1^e (\ln t + 1)\,dt = \Big[(t \ln t - t) + t\Big]_1^e$$

$$= \Big[t \ln t\Big]_1^e = e \cdot 1 - 1 \cdot 0 = e$$

[9.8] 34.

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34. $\underline{F} = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}$ $\qquad C: \left.\begin{aligned} x &= t^3 \\ y &= t^2 \\ z &= t \end{aligned}\right\} t \in [1, 3]$

$dx = 3t^2 dt$
$dy = 2t\,dt$
$dz = dt$

$$W = \int_C \underline{F} \cdot d\underline{r} = \int_C yz\,dx + xz\,dy + xy\,dz$$

$$= \int_1^3 (t^2)(t)\,3t^2 dt + (t^3)(t)\,2t\,dt + (t^3)(t^2)\,dt$$

$$= \int_1^3 6t^5 dt = 6\left[\frac{t^6}{6}\right]_1^3 = 3^6 - 1 = 728$$

ABC trial 4

[9.9] 03.

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3. $\frac{\partial}{\partial y}(x+2y) - \frac{\partial}{\partial x}(2x-y) = 2 - 2 = 0$

$$\varphi = \int (x+2y)\,dx = \frac{x^2}{2} + 2xy + g(y)$$

$$\varphi = \int (2x-y)\,dy = 2xy - \frac{y^2}{2} + f(x)$$

$$\varphi = \frac{x^2 - y^2}{2} + 2xy + C$$

$$\int_{(1,0)}^{(3,2)} (x+2y)\,dx + (2x-y)\,dy = \varphi(3,2) - \varphi(1,0)$$

$$= \frac{3^2 - 2^2}{2} + 2\cdot 3\cdot 2 - \left(\frac{1^2 - 0^2}{2} + 0\right) = 14$$

[9.9] 05.

5. $\frac{\partial}{\partial y}\left(\frac{-y}{y^2}\right) - \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right) = \frac{1}{y^2} - \frac{1}{y^2} = 0$

$\varphi = \int -\frac{1}{y}\,dx = -\frac{x}{y} + g(y)$

$\varphi = \int +\frac{x}{y^2}\,dy = -\frac{x}{y} + h(x)$, $\quad \varphi = -\frac{x}{y} + C$

$$\int_{(4,1)}^{(4,4)} \frac{-y\,dx + x\,dy}{y^2} = \varphi(4,4) - \varphi(4,1)$$

$$= -\frac{4}{4} - \left(-\frac{4}{1}\right) = 3$$

[9.9] 13.

13. $\frac{\partial}{\partial y}\left(y^2\cos(xy^2)\right) - \frac{\partial}{\partial x}\left(-2xy\sin(xy^2)\right)$

$= 2y^2\cos(xy^2) - 2xy^3\sin(xy^2)$

$\quad - \left[-2y\sin(xy^2) - 2y^3\cos(xy^2)\right] \neq 0$

not an exact differential

[9.9] 14.

14. $P_y = -4xy(x^2+y^2+1)^{-3} = Q_x$ and the vector field is a gradient field.
$\phi_x = x(x^2+y^2+1)^{-2}$, $\phi = -\frac{1}{2}(x^2+y^2+1)^{-1} + g(y)$, $\phi_y = y(x^2+y^2+1)^{-2} + g'(y) = y(x^2+y^2+1)^{-2}$,
$g(y) = 0$, $\phi = -\frac{1}{2}(x^2+y^2+1)^{-1}$

[9.9] 17.

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17. $\frac{\partial}{\partial y}\left(2x+e^{-y}\right)-\frac{\partial}{\partial x}\left(4y-xe^{-y}\right)=-e^{-y}-\left(-e^{-y}\right)=0$

$\phi=\int\left(2x+e^{-y}\right)dx=x^2+xe^{-y}+g(y)$

$\phi=\int\left(4y-xe^{-y}\right)dy=\frac{4y^2}{2}+xe^{-y}+h(x)$

$\phi=x^2+2y^2+xe^{-y}+C$

$\int_C\left(2x+e^{-y}\right)dx+\left(4y-xe^{-y}\right)dy=\int_{(0,0)}^{(1,1)}d\phi=\phi(1,1)-\phi(0,0)$

$=1^2+2(1^2)+1e^{-1}-0=3+e^{-1}=3.3678$

[9.9] 18.

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18. Since $P_y=-e^{-y}=Q_x$, $\mathbf{F}$ is conservative and $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path. Thus, instead of the given curve we may use the simpler curve C_1: $y=0$, $-2\le -x\le 2$. Then $dy=0$ and

$$W=\int_{C_1}(2x+e^{-y})\,dx+(4y-xe^{-y})\,dy=\int_2^{-2}(2x+1)\,dx=(x^2+x)\Big|_2^{-2}=(4-2)-(4+2)=-4.$$

[9.9] 22.

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22. $P_y=0=Q_x$, $Q_z=0=R_y$, $R_x=0=P_z$, and the integral is independent of path. Parameterize the line segment between the points by $x=1+2t$, $y=2+2t$, $z=1$, for $0\le t\le 1$. Then $dx=2\,dt$, $dz=0$ and

$$\int_{(1,2,1)}^{(3,4,1)}(2x+1)\,dx+3y^2\,dy+\frac{1}{z}\,dz=\int_0^1[(2+4t+1)2+3(2+2t)^2 2]\,dt$$

$$=\int_0^1(24t^2+56t+30)\,dt=(8t^3+28t^2+30t)\Big|_0^1=66.$$

[9.9] 25.

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25. $P_y=1-z\sin x=Q_x$, $Q_z=\cos x=R_y$, $R_x=-y\sin x=P_z$ and the integral is independent of path. Integrating $\phi_x=y-yz\sin x$ we find $\phi=xy+yz\cos x+g(y,z)$. Then $\phi_y=x+z\cos x+g_y(y,z)=Q=x+z\cos x$, so $g_y=0$, $g(y,z)=h(z)$, and $\phi=xy+yz\cos x+h(z)$. Now $\phi_z=y\cos x+h(z)=R=y\cos x$, so $h(z)=0$ and $\phi=xy+yz\cos x$. Since $\mathbf{r}(0)=4\mathbf{j}$ and $\mathbf{r}(\pi/2)=\pi\mathbf{i}+\mathbf{j}+4\mathbf{k}$,

$$\int_C \mathbf{F}\cdot d\mathbf{r}=(xy+yz\cos x)\Big|_{(0,4,0)}^{(\pi,1,4)}=(\pi-4)-(0+0)=\pi-4.$$