TUT 4: [9.5] 3, 4, 6, 8, 15, 25, 29, 39, 47. [9.6] 6, 8, 10, 13, 19, 22, 25, 31, 33. [9.7] 1, 5, 6, 8, 13, 17, 18, 23, 27, 28.

[9.5] 03.

3. $\underline{\nabla} F = \left(\frac{y^2}{z^3}\right)\underline{i} + \left(\frac{2xy}{z^3}\right)\underline{j} + \left(-\frac{3xy^2}{z^4}\right)\underline{k}$

[9.5] 04.

4. $F = xy\cos yz$

$$\underline{\nabla} F = \underline{i}\left(y\cos yz\right) + \underline{j}\left(x\cos(yz) - xyz\sin(yz)\right) + \underline{k}\left(-xy^2\sin(yz)\right)$$

[9.5] 06.

6. $$\underline{\nabla} f = \frac{3x^2y}{2\sqrt{x^3y-y^4}}\,\underline{i} + \frac{x^3-4y^3}{2\sqrt{x^3y-y^4}}\,\underline{j}$$

$$\underline{\nabla} f\Big|_{\substack{x=3\\y=2}} = \frac{3(3^2)(2)}{2\sqrt{3^3\cdot 2-2^4}}\,\underline{i} + \frac{(3)^3-4(2)^3}{2\sqrt{3^3\cdot 2-2^4}}\,\underline{j}$$

$$= \frac{27}{\sqrt{38}}\,\underline{i} - \frac{5}{2\sqrt{38}}\,\underline{j} = 4.38\,\underline{i} - 0.406\,\underline{j}$$

[9.5] 08.

8. $$\underline{\nabla} F\Big|_{\substack{x=-4\\y=3\\z=5}} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \frac{1}{x^2+y^2+z^2}\Big|_{\substack{x=-4\\y=3\\z=5}} = \frac{1}{50}\begin{bmatrix} -8 \\ 6 \\ 10 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix}$$

[9.5] 15.

15. $$\underline{u} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Normalize $\underline{u}$:

$$\underline{u} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\underline{\nabla} f = \begin{bmatrix} 2(xy+1)y \\ 2(xy+1)x \end{bmatrix} \qquad \underline{\nabla} f\Big|_{(x,y)=(3,2)} = \begin{bmatrix} 2\cdot 7\cdot 2 \\ 2\cdot 7\cdot 3 \end{bmatrix}$$

$$\underline{\nabla} f\Big|_{(3,2)} = \begin{bmatrix} 28 \\ 42 \end{bmatrix}$$

$$D_{\underline{u}} f = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 28 \\ 42 \end{bmatrix} = \frac{98}{\sqrt{5}} = 43.83$$

[9.5] 25.

25. $F(x,y,z) = x^2 + 4xz + 2yz^2$

$$\underline{\nabla} F = (2x+4z)\underline{i} + (2z^2)\underline{j} + (4x+4yz)\underline{k}$$

Direction of most rapid increase at $[1,2,-1]$ is

$$\underline{\nabla} F \Big|_{\substack{x=1\\y=2\\z=-1}} = (2(1)+4(-1))\underline{i} + (2(-1)^2)\underline{j} + (4(1)+4(2)(-1))\underline{k}$$

$$= -2\underline{i} + 2\underline{j} - 4\underline{k}$$

Maksimum rate $= \|\underline{\nabla} F\|$

$$\|\underline{\nabla} F\| = \sqrt{(-2)^2 + (2)^2 + (-4)^2} = \sqrt{24} = 4.899$$

[9.5] 29.

29. $F(x,y,z) = e^y \sqrt{xz}$

$$\underline{\nabla} F = \frac{ze^y}{2\sqrt{xz}}\underline{i} + e^y\sqrt{xz}\,\underline{j} + \frac{xe^y}{2\sqrt{xz}}\underline{k}$$

$$\underline{\nabla} F \Big|_{\substack{x=16\\y=0\\z=9}} = \frac{9 \cdot 1}{2\sqrt{16 \cdot 9}}\underline{i} + 1 \cdot \sqrt{16 \cdot 9}\,\underline{j} + \frac{16 \cdot 1}{2\sqrt{16 \cdot 9}}\underline{k}$$

$$= \frac{3}{8}\underline{i} + 12\underline{j} + \frac{2}{3}\underline{k}$$

Direction of most rapid decrease is $-\underline{\nabla} F$,

i.e. $-\frac{3}{8}\underline{i} - 12\underline{j} - \frac{2}{3}\underline{k}$

Magnitude of rate $= \|\underline{\nabla} F\|$

$$\|\underline{\nabla} F\| = \sqrt{\left(-\frac{3}{8}\right)^2 + (-12)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{144\frac{337}{576}} = 12.024$$

[9.5] 39.

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39. $T(x,y) = 5 + 2x^2 + y^2$

$$\underline{\nabla} T = 4x\,\underline{i} + 2y\,\underline{j}$$

$$\underline{\nabla} T \Big|_{\substack{x=4\\ y=2}} = 4(4)\underline{i} + 2(2)\underline{j} = 16\underline{i} + 4\underline{j}$$

Most rapid decrease is in direction $-\underline{\nabla} T$

$= -16\underline{i} - 4\underline{j}$ or unit vector $=$

$-0.97\underline{i} - 0.24\underline{j}$

[9.5] 47.

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47. $$\underline{\nabla}(fg) = \underline{i}\frac{\partial}{\partial x}(fg) + \underline{j}\frac{\partial}{\partial y}(fg) + \underline{k}\frac{\partial}{\partial z}(fg)$$

$$= \underline{i}\left[\frac{\partial f}{\partial x}\cdot g + f\cdot\frac{\partial g}{\partial x}\right] + \underline{j}\left[\frac{\partial f}{\partial y}\cdot g + f\cdot\frac{\partial g}{\partial y}\right]$$

$$+ \underline{k}\left[\frac{\partial f}{\partial z}\cdot g + f\cdot\frac{\partial g}{\partial z}\right]$$

$$= \left[\underline{i}\frac{\partial f}{\partial x} + \underline{j}\frac{\partial f}{\partial y} + \underline{k}\frac{\partial f}{\partial z}\right] g$$

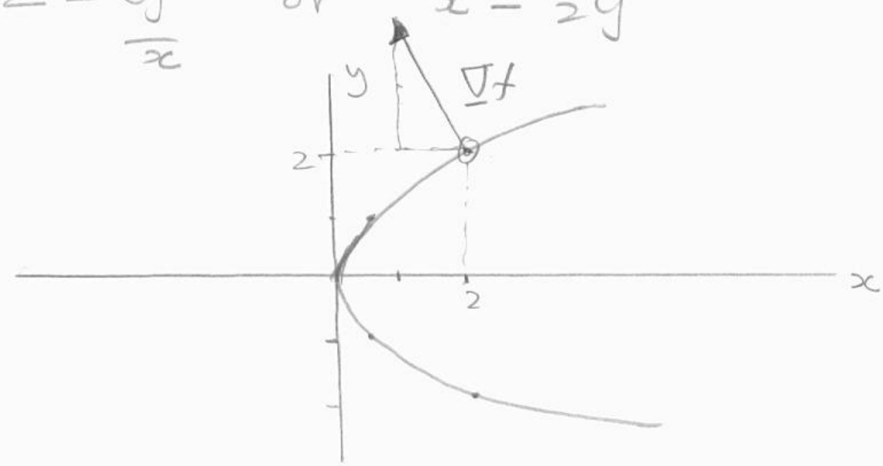
$$+ f\left[\underline{i}\frac{\partial g}{\partial x} + \underline{j}\frac{\partial g}{\partial y} + \underline{k}\frac{\partial g}{\partial z}\right].$$

$$= g\,\underline{\nabla} f + f\,\underline{\nabla} g$$

[9.6] 06.

6. $f(x,y) = \dfrac{y^2}{x}$, $f(2,2) = \dfrac{4}{2} = 2$

$$2 = \frac{y^2}{x} \quad \text{or} \quad x = \tfrac{1}{2}y^2$$



$$\nabla f = -\frac{y^2}{x^2}\,\hat{i} + \frac{2y}{x}\,\hat{j}$$

$$\nabla f \Big|_{\substack{x=2\\y=2}} = -\frac{4}{4}\,\hat{i} + \frac{4}{2}\,\hat{j} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

[9.6] 08.

8. $f(x,y) = \dfrac{y-1}{\sin x}$ at $x = \frac{\pi}{6}$, $y = \frac{3}{2}$.

Level curve: $\dfrac{y-1}{\sin x} = \dfrac{\frac{3}{2}-1}{\sin(\frac{\pi}{6})} = 1$

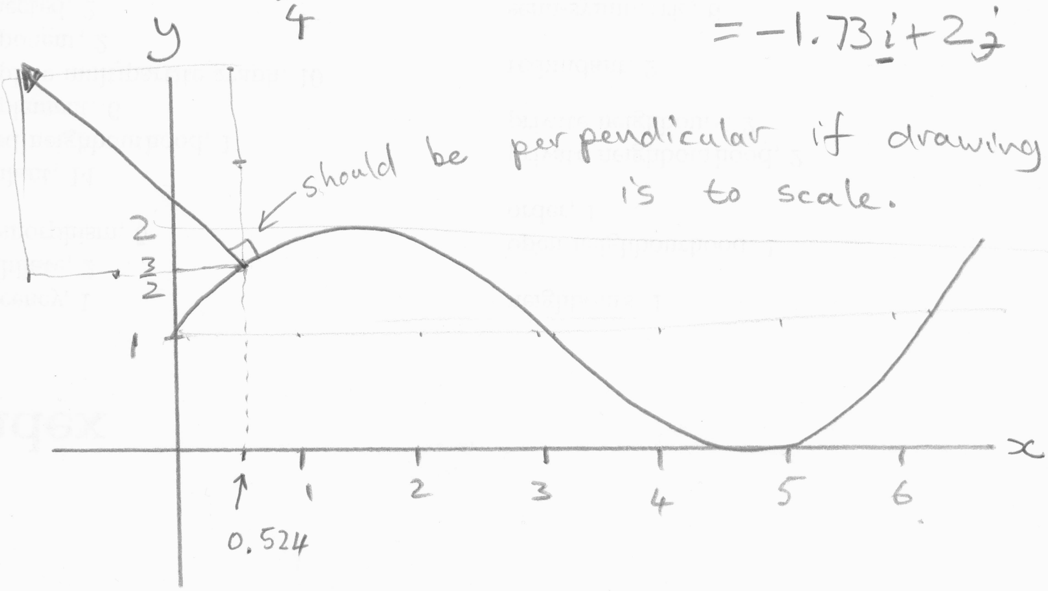
or $y = \sin x + 1$

$$\underline{\nabla} f = \frac{\cos x\,(y-1)}{\sin^2 x}\,\hat{\underline{i}} + \frac{1}{\sin x}\,\hat{\underline{j}}$$

$$\underline{\nabla} f \Big|_{\substack{x=\pi/6 \\ y=3/2}} = \frac{-\cos(\frac{\pi}{6})\cdot(\frac{3}{2}-1)}{\left(\sin(\frac{\pi}{6})\right)^2}\,\hat{\underline{i}} + \frac{1}{\sin(\frac{\pi}{6})}\,\hat{\underline{j}}$$

$$= \frac{-\frac{\sqrt{3}}{2}\cdot\frac{1}{2}}{\frac{1}{4}}\,\hat{\underline{i}} + \frac{1}{\frac{1}{2}}\,\hat{\underline{j}} = -\sqrt{3}\,\underline{i} + 2\,\underline{j}$$

$$= -1.73\,\underline{i} + 2\,\underline{j}$$



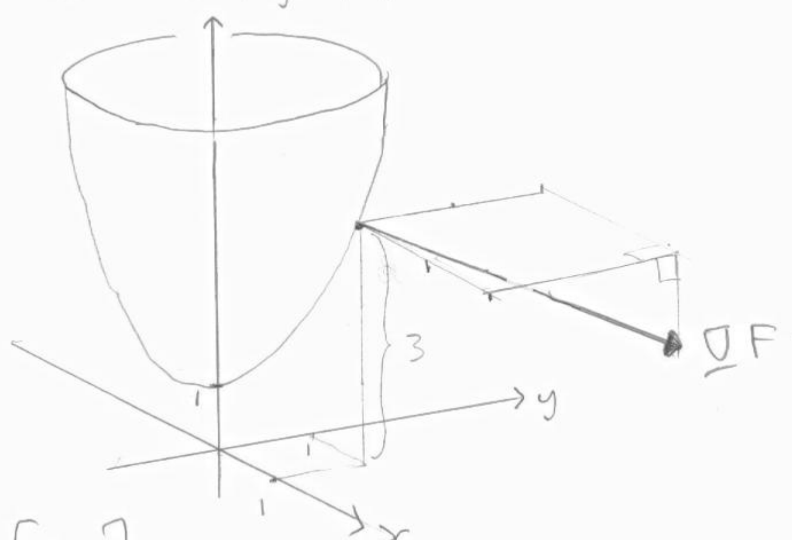
[9.6] 10.

10. $F(x,y,z) = x^2+y^2-z$

$F(1,1,3) = 1^2+1^3-3 = -1$

Level surface: $x^2+y^2-z = -1$

or $z = x^2+y^2+1$



$$\underline{\nabla} F = \begin{bmatrix} 2x \\ 2y \\ -1 \end{bmatrix} \qquad \underline{\nabla} F \Big|_{\substack{x=1 \\ y=1 \\ z=3}} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

[9.6] 13.

13. Surface given by

$$F(x,y,z) = x^2+y^2-z = 0$$

$$\underline{\nabla} F = 2x\underline{i} + 2y\underline{j} - \underline{k}$$

scaling factor.

Let $2x\underline{i} + 2y\underline{j} - \underline{k} = \alpha\left(4\underline{i} + \underline{j} + \frac{1}{2}\underline{k}\right)$

$$-1 = \tfrac{1}{2}\alpha \longrightarrow \alpha = -2.$$

$$2x = -2(4) \longrightarrow x = -4$$

$$2y = -2(1) \longrightarrow y = -1$$

$$z = x^2+y^2 = (-4)^2+(-1)^2 = 17$$

point is at $-4\underline{i} - \underline{j} + 17\underline{k}$

[9.6] 19.

19. $F(x,y,z) = 25 - x^2 - y^2 - z$

Surface is $25 - x^2 - y^2 - z = 25 - (3)^2 - (-4)^2 - 0$
$= 0$

$\underline{\nabla} F = -2x\underline{i} - 2y\underline{j} - \underline{k}$, $\underline{\nabla} F \Big|_{\substack{x=3\\y=-4\\z=0}} = -6\underline{i} + 8\underline{j} - \underline{k}$

$\underline{r}_0 = 3\underline{i} - 4\underline{j}$

Eq. of tangent plane $\underline{n} \cdot \underline{r} = \underline{n} \cdot \underline{r}_0$.

$-6x + 8y - z = \begin{bmatrix} -6 \\ 8 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = -50$

$-6x + 8y - z = -50$ or $6x - 8y + z = 50$

[9.6] 22.

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22. $x^2y^3 + 6z = 0$

$$F(x,y,z) = x^2y^3 + 6z$$

$$\underline{\nabla} F = \begin{bmatrix} 2xy^3 \\ 3x^2y^2 \\ 6 \end{bmatrix} \qquad \underline{\nabla} F \Big|_{\substack{x=2 \\ y=1 \\ z=1}} = \begin{bmatrix} 4 \\ 12 \\ 6 \end{bmatrix}$$

$$\underline{n} = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix} \qquad \underline{r}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad \underline{n} \cdot \underline{r}_1 = 4 + 6 + 3 = 13$$

Eq. of tangent plane through $\underline{r}_1$:

$$2x + 6y + 3z = 13$$

[9.6] 25.

25. $\underline{n}$ of tangent plane is $2\underline{i}+4\underline{j}+6\underline{k}$

(or $-2\underline{i}-4\underline{j}-6\underline{k}$)

$F(x,y,z) = x^2+y^2+z^2$

$\underline{\nabla}F = 2x\underline{i}+2y\underline{j}+2z\underline{k}$

Find x, y, z so that

$$2x\underline{i}+2y\underline{j}+2z\underline{k} = (2\underline{i}+4\underline{j}+6\underline{k})\alpha.$$

$x=1\alpha, \quad y=2\alpha, \quad z=3\alpha$

but $x^2+y^2+z^2=7=\alpha^2+4\alpha^2+9\alpha^2=14\alpha^2.$

therefore $14\alpha^2=7$ or $\alpha=\pm\frac{1}{\sqrt{2}}.$

Points on surface: $\underline{r}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\2\\3\end{bmatrix}$ or $\underline{r}=-\frac{1}{\sqrt{2}}\begin{bmatrix}1\\2\\3\end{bmatrix}$

[9.6] 31.

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31. $F = x^2+y^2-z^2$

$\underline{\nabla} F = 2x\,\underline{i} + 2y\,\underline{j} - 2z\,\underline{k} = \underline{n}.$

$\underline{r}_0 = x\,\underline{i} + y\,\underline{j} + \sqrt{x^2+y^2}\,\underline{k}$

General tangent plane at $[x_0, y_0, z_0]$ is.

$$2x_0 x + 2y_0 y - 2z_0 z = \begin{bmatrix} 2x_0 \\ 2y_0 \\ -2z_0 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = 2x_0^2 + 2y_0^2 - 2z_0^2$$

$$= 2x_0^2 + 2y_0^2 - 2\sqrt{x_0^2+y_0^2} = 0$$

Eq. of general tangent plane $x_0 x + y_0 y - z_0 z = 0$

if $x=0$, and $y=0$, then $z=0 \rightarrow$ passes through origin.

[9.6] 33.

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33. $F(x,y,z) = x^2 + 2y^2 + z^2$

$\underline{\nabla} F = 2x\,\underline{i} + 4y\,\underline{j} + 2z\,\underline{k}$

$$\underline{\nabla} F \Big|_{\substack{x=1 \\ y=-1 \\ z=1}} = 2(1)\underline{i} + 4(-1)\underline{j} + 2(1)\underline{k}$$

$$= 2\underline{i} - 4\underline{j} + 2\underline{k} = \underline{a}$$

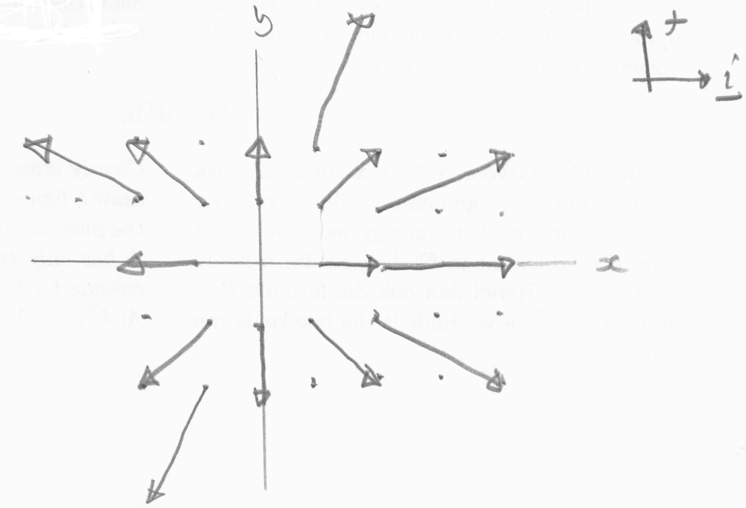
$\underline{r}_0 = \underline{i} - \underline{j} + \underline{k}$

$$\underline{r} = \underline{r}_0 + t\underline{a} = \underline{i} - \underline{j} + \underline{k} + t(2\underline{i} - 4\underline{j} + 2\underline{k})$$

$$= \begin{bmatrix} 1+2t \\ -1-4t \\ 1+2t \end{bmatrix} \quad \text{or} \quad \begin{aligned} x &= 1+2t \\ y &= -1-4t, \\ z &= 1+2t \end{aligned}$$

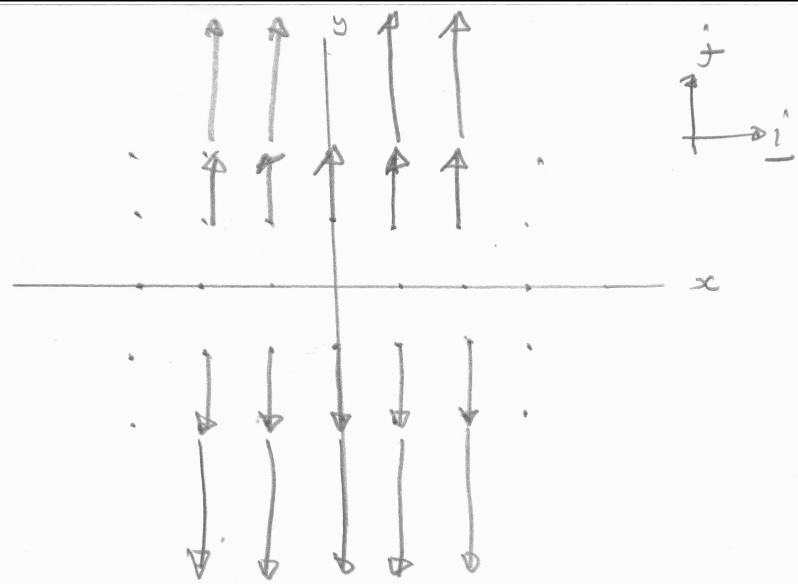
[9.7] 01.

1. $\underline{F} = x\underline{i} + y\underline{j}$



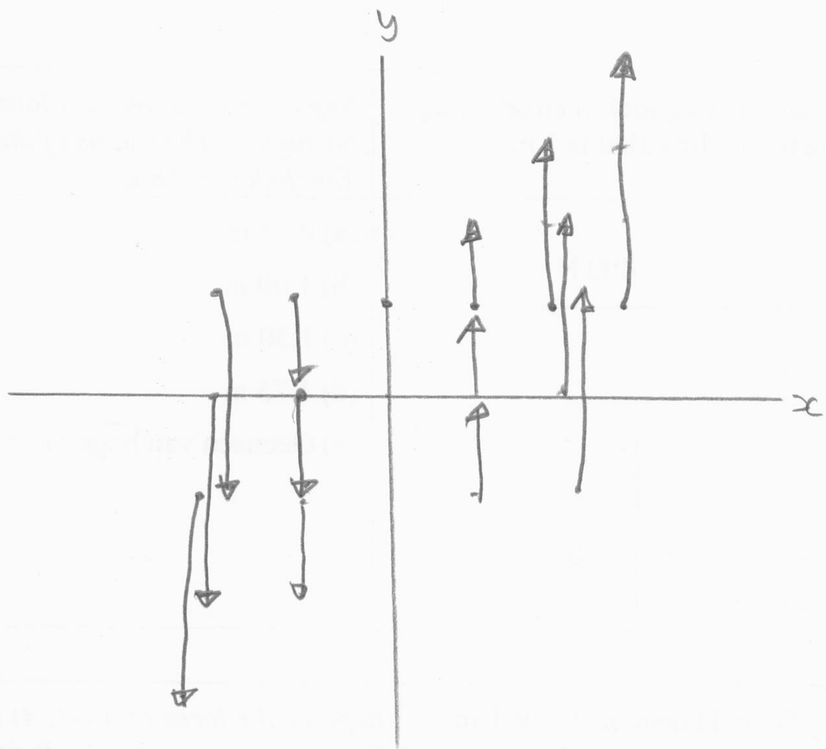
[9.7] 05.

5. $\underline{F} = y\underline{j}$



[9.7] 06.

6.



[9.7] 08.

8. $\underline{F} = \begin{bmatrix} 10yz \\ 2x^2z \\ 6x^3 \end{bmatrix}$ $\quad \underline{\nabla} \cdot \underline{F} = 0 + 0 + 0 = 0$

$$\underline{\nabla} \times \underline{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} 10yz \\ 2x^2z \\ 6x^3 \end{bmatrix} = \begin{bmatrix} 0 - 2x^2 \\ 10y - 18x^2 \\ 4xz - 10z \end{bmatrix}$$

[9.7] 13.

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3. $\underline{\nabla}\cdot\underline{F} = \frac{\partial}{\partial x}\left(xe^{-z}\right) + \frac{\partial}{\partial y}\left(4yz^2\right) + \frac{\partial}{\partial z}\left(3ye^{-z}\right)$

$$= e^{-z} + 4z^2 - 3ye^{-z} = e^{-z}(1-3y) + 4z^2$$

$$\underline{\nabla}\times\underline{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} xe^{-z} \\ 4yz^2 \\ 3ye^{-z} \end{bmatrix} = \begin{bmatrix} 3e^{-z} - 8yz \\ -xe^{-z} - 0 \\ 0 - 0 \end{bmatrix}$$

$$= (3e^{-z} - 8yz)\underline{i} - xe^{-z}\underline{j}$$

[9.7] 17.

17.

$$\underline{\nabla}\cdot\underline{r} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 1+1+1 = 3$$

[9.7] 18.

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18. $\underline{\nabla}\times\underline{r} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \underline{0}$

[9.7] 23.

$$23.\quad \underline{\nabla} \times ((\underline{r}\cdot\underline{r})\,\underline{a}) = \underline{\nabla} \times ((x^2+y^2+z^2)\,\underline{a})$$

$$= \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} (x^2+y^2+z^2)a_1 \\ (x^2+y^2+z^2)a_2 \\ (x^2+y^2+z^2)a_3 \end{bmatrix} = \begin{bmatrix} 2ya_3 - 2za_2 \\ 2za_1 - 2xa_3 \\ 2xa_2 - 2ya_1 \end{bmatrix}$$

$$= 2\begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 2\,\underline{r} \times \underline{a}$$

[9.7] 27.

$$27.\quad \underline{\nabla}\cdot(f\,\underline{F}) \qquad \underline{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$$

$$= \frac{\partial}{\partial x}(f F_1) + \frac{\partial}{\partial y}(f F_2) + \frac{\partial}{\partial z}(f F_3)$$

$$= \frac{\partial f}{\partial x}F_1 + f\frac{\partial F_1}{\partial x} + \frac{\partial f}{\partial y}F_2 + f\frac{\partial F_2}{\partial y} + \frac{\partial f}{\partial z}F_3 + f\frac{\partial F_3}{\partial z}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} + f\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$$

$$= (\underline{\nabla} f)\cdot\underline{F} + f(\underline{\nabla}\cdot\underline{F})$$

[9.7] 28.

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28. $\underline{\nabla} \times (f \underline{F})$

$$= \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} fP \\ fQ \\ fR \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial y}(fR) - \frac{\partial}{\partial z}(fQ) \\ \frac{\partial}{\partial z}(fP) - \frac{\partial}{\partial x}(fR) \\ \frac{\partial}{\partial x}(fQ) - \frac{\partial}{\partial y}(fP) \end{bmatrix}$$

$$= \begin{bmatrix} f_y R + f R_y - f_z Q - f Q_z \\ f_z P + f P_z - f_x R - f R_x \\ f_x Q + f Q_x - f_y P - f P_y \end{bmatrix}$$

$$= \begin{bmatrix} f_y R - f_z Q \\ f_z P - f_x R \\ f_x Q - f_y P \end{bmatrix} + \begin{bmatrix} f R_y - f Q_z \\ f P_z - f R_x \\ f Q_x - f P_y \end{bmatrix}$$

$$= \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} + f \cdot \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$