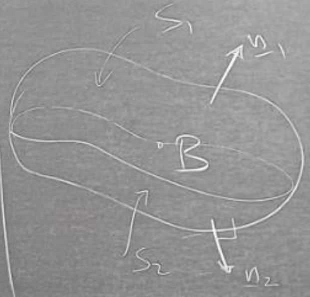
Appl.Maths. B242-2023: LECTURE 23

LECTURE 23 [4.16] DIVERGENCE THEOREM

1

"GAUSS' TH"

Flux out of B



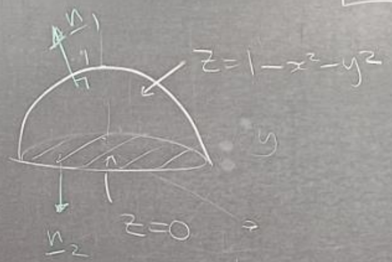
$$\boxed{\iint_S \underline{E}\cdot\underline{n}\, dS = \iiint_B \underline{\nabla}\cdot\underline{E}\, dV}$$

$\underbrace{\iint_S \underline{E}\cdot\underline{n}\, dS}_{\text{Flux out of B, over S}}$ $\underbrace{\iiint_B \underline{\nabla}\cdot\underline{E}\, dV}_{\sum \frac{\text{flux}}{\text{volume}} \times \text{volume}}$

Example Verify Gauss

2

$$\underline{E} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



LHS of Gauss:

$$\iint_S \underline{E}\cdot\underline{n}\, dS = \iint_{S_1} + \iint_{S_2}$$

$\iint_{S_1}$

Normal: $z = f(x,y) = 1 - x^2 - y^2$

$G = z + x^2 + y^2 - 1$

$$\underline{\nabla} G = \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix}, \quad \underline{n}_1 = \frac{1}{\sqrt{1+4x^2+4y^2}} \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix}$$

Surf.-elem.:

$$dS = \sqrt{1 + (-2x)^2 + (-2y)^2}\, dA$$

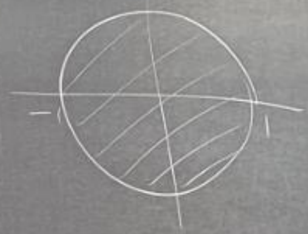
$$= \sqrt{1+4x^2+4y^2}\, dA$$

$\boxed{3}$

$$\iint_{S_1} = \iint_R \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix} dA$$

$z = 1 - x^2 - y^2$

Region



$$= \iint_R \left(2x^2 + 2y^2 + (1 - x^2 - y^2)\right) dA$$

$$= \iint_R (x^2 + y^2 + 1)\, dA$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (r^2 + 1)\, r\, dr\, d\theta$$

$$= \left[\theta\right]_0^{2\pi} \cdot \left[\frac{r^4}{4} + \frac{r^2}{2}\right]_0^1 = 2\pi\left(\frac{1}{4} + \frac{1}{2}\right) = 2\pi\,\frac{3}{4} = \frac{3\pi}{2}$$

$\boxed{4}$

$\iint_{S_2}$ Normal $\underline{n} = -\underline{k} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

Surface element

$dS = 1\, dA$

$z = 0$

$G = z$

$-\nabla G = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$,

$$\iint_{S_2} = \iint_R \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} dA$$

$z = 0$

$$= \iint_R 0\, dA = 0$$

Flux out $= \frac{3\pi}{2} + 0 = \frac{3\pi}{2}$

RHS:

$$\underline{\nabla}\cdot\underline{F} = \begin{bmatrix}\partial_x\\ \partial_y\\ \partial_z\end{bmatrix}\cdot\begin{bmatrix}x\\ y\\ z\end{bmatrix} = 3$$

$$\text{Flux out} = \iiint_B 3\,dV = 3\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{1-r^2} dz\, r\,dr\,d\theta$$

$$= 3\Big[\theta\Big]_0^{2\pi}\int_{r=0}^{1}\Big[z\Big]_0^{1-r^2} r\,dr$$

$$= 3(2\pi)\int_0^1 (r-r^3)\,dr$$

$$= 6\pi\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 = \frac{6\pi}{4} = \frac{3\pi}{2} \leftarrow$$

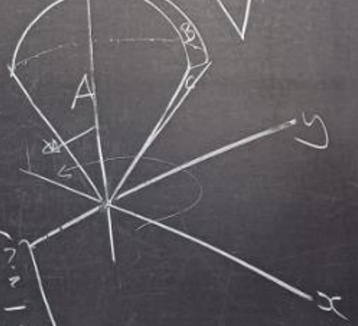
Example 2: Use Gauss to calc. flux out of B.

$$\underline{F} = -xz\,\underline{i} + y^2\,\underline{j} + z^2\,\underline{k}$$

$$B: \begin{cases} z \geq \sqrt{x^2+y^2} & \leftarrow \text{cone, top part, above} \\ x^2+y^2+z^2 \leq 4 & \leftarrow \text{sphere, rad} = 2\text{, inside} \\ y \geq 0 & \leftarrow \end{cases}$$

$\longrightarrow y$ only right half.

Calc. $P = \iint_S \underline{F}\cdot\underline{n}\,dS$



LHS:

$\underline{n} = -\underline{j}$

$\underline{n} = \begin{bmatrix}2x\\ 2y\\ 2z\end{bmatrix}\frac{1}{\ldots}$

$\underline{n} = \begin{bmatrix}?\\ ?\\ -\end{bmatrix}$

$\boxed{7}$

RHS: $\nabla \cdot F = \nabla \cdot \begin{bmatrix} -xz \\ yz \\ z^2 \end{bmatrix} = -z + z + 2z = 2z$

$$\text{Flux out} = \iiint_B 2z \, dV$$

Do in spherical:

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{2} (2\rho \cos\phi)(\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta)$$

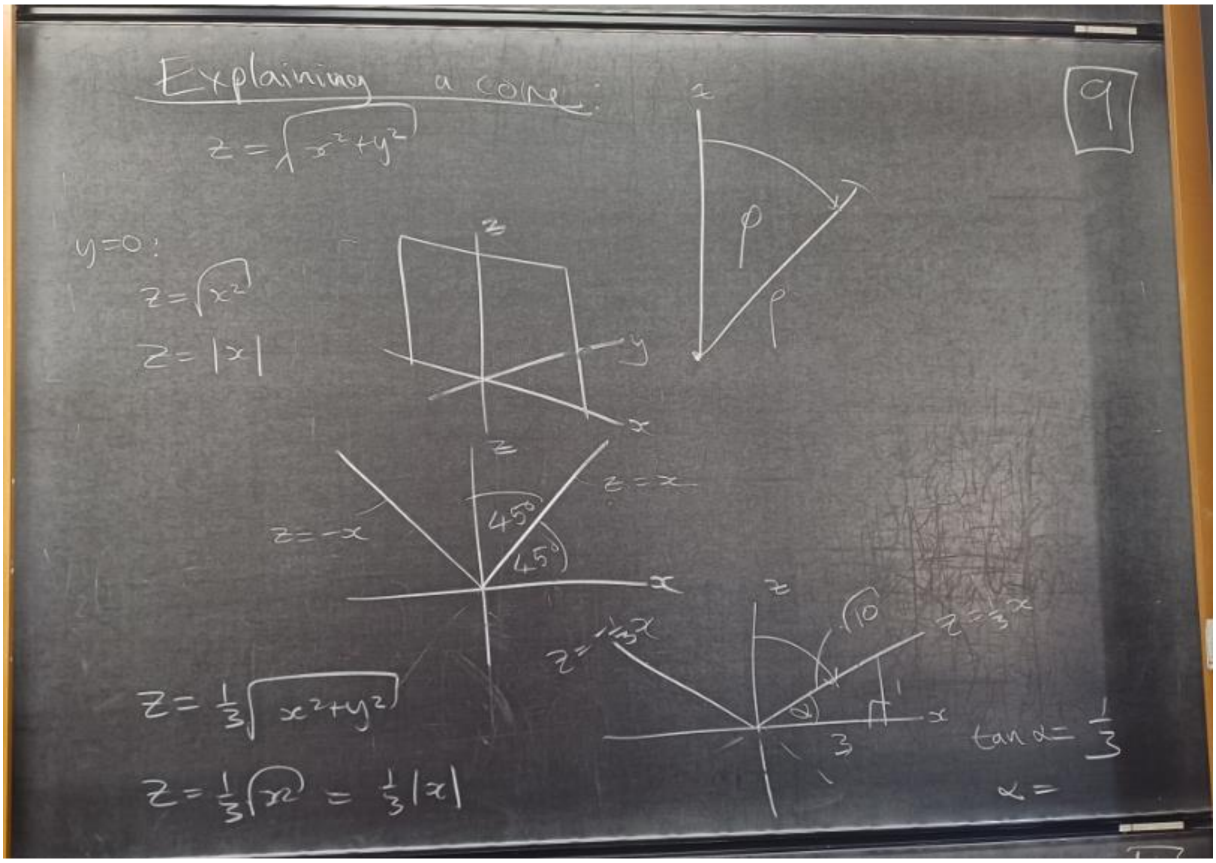
$$= \left[\theta\right]_0^{\pi} \cdot \int_{\phi=0}^{\pi/4} 2\cos\phi \sin\phi \, d\phi \cdot \int_0^2 \rho^3 \, d\rho$$

$\boxed{8}$

$$= \pi \int_0^{\pi/4} \sin 2\phi \, d\phi \left[\frac{\rho^4}{4}\right]_0^2$$

$$= \pi \left[\frac{-\cos(2\phi)}{2}\right]_0^{\pi/4} \frac{16}{4}$$

$$= \frac{4\pi}{2}\left[0 - (-1)\right] = 2\pi$$


Explaining a cone:
$z = \sqrt{x^2+y^2}$
9
$y=0$:
$z=\sqrt{x^2}$
$z=|x|$
ρ
ρ
$z=x$
$z=-x$
45°
45°
$\sqrt{10}$
$z=\frac{1}{3}x$
$z=-\frac{1}{3}x$
α
3
1
$\tan\alpha = \frac{1}{3}$
$\alpha =$
$z = \frac{1}{3}\sqrt{x^2+y^2}$
$z = \frac{1}{3}\sqrt{x^2} = \frac{1}{3}|x|$