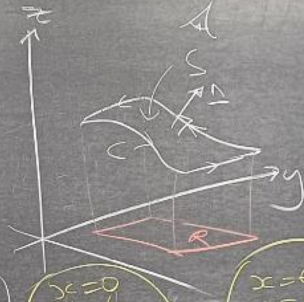
Appl.Maths. B242-2023: LECTURE 19

LECTURE 19 [9.14] STOKES

1

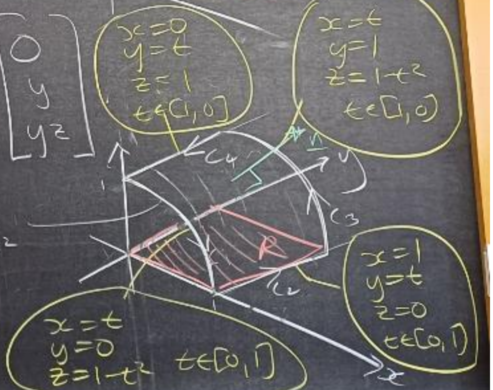
$$\oint_C \underline{F}\cdot d\underline{r} = \iint_S (\underline{\nabla}\times\underline{F})\cdot\underline{n}\, dS$$



Example: Verify Stokes

$$S: \begin{cases} z = 1 - x^2 \\ y \in [0,1] \\ z \geq 0 \\ x \geq 0 \end{cases} \qquad \underline{F} = \begin{bmatrix} 0 \\ y \\ yz \end{bmatrix}$$

$z = 1 - x^2$



STOKES LHS

$$W = \oint_C \underline{F}\cdot d\underline{r} = \oint_C y\,dy + yz\,dz$$

2

$$W = \underbrace{\int_0^1 0}_{C_1} + \underbrace{\int_0^1 t\,dt}_{C_2} + \underbrace{\int_1^0 1\cdot 0 + 1(1-t^2)(-2t\,dt)}_{C_3} + \underbrace{\int_1^0 t\,dt + t(0)}_{C_4}$$

$$= \cancel{\int_0^1 t\,dt} + \int_1^0 (-2t + 2t^3)\,dt + \cancel{\int_1^0 t\,dt}$$

$$= \left[-\frac{2t^2}{2} + \frac{2t^4}{4}\right]_1^0 = 0 - \left(-1 + \frac{1}{2}\right) = \frac{1}{2}$$

$\boxed{\text{Stokes RHS}}$ $\quad W = \iint_S (\nabla \times \underline{F}) \cdot \underline{n} \, dS$ $\quad \boxed{3}$

Curl

$$\underline{\nabla} \times \underline{F} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} 0 \\ y \\ yz \end{bmatrix} = \begin{bmatrix} z - 0 \\ 0 - 0 \\ 0 - 0 \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}$$

Normal

$S: z = 1 - x^2$

$G = 1 - x^2 - z$

$$-\underline{\nabla} G = \begin{bmatrix} +2x \\ 0 \\ +1 \end{bmatrix}$$

$$\underline{n} = \frac{1}{\sqrt{1 + (2x)^2}} \begin{bmatrix} 2x \\ 0 \\ 1 \end{bmatrix}$$

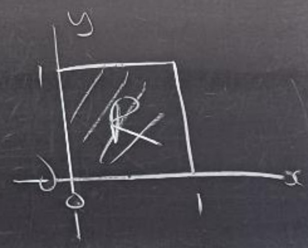
Surf-elem

$z = 1 - x^2$

$$dS = \sqrt{1 + (-2x)^2 + (0)^2} \, dA$$

$$= \sqrt{1 + 4x^2} \, dA$$

Region



$$W = \iint_R \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2x \\ 0 \\ 1 \end{bmatrix} \cancel{\frac{\sqrt{1+4x^2}}{\sqrt{1+(2x)^2}}} \, dA \qquad \boxed{4}$$

$$= \iint_R (1 - x^2) 2x \, dA = \int_{y=0}^{1} dy \int_{x=0}^{1} (2x - 2x^3) \, dx \, \cancel{dy}$$

$$= 1 \cdot \left[\frac{2x^2}{2} - \frac{2x^4}{4} \right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

Stokes is verified

$\iint$ in ~~polar~~ $\quad \cancel{\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ dA &= r\,dr\,d\theta \end{aligned}}}$

parameterizing

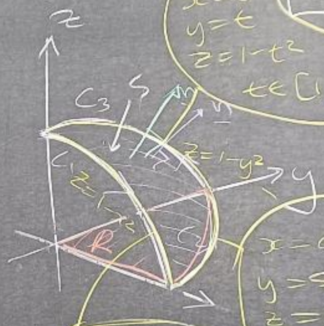
$x = 1 \cos t$

$y = 1 \sin t$ $\quad t \in [0, \frac{\pi}{2}]$

$\boxed{5}$

Example: Verify Stokes.

$$S: \begin{cases} z = 1 - x^2 - y^2 \\ x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{cases} \qquad \underline{F} = \begin{bmatrix} zy \\ x \\ 1 \end{bmatrix}$$



$x = 0$, $y = t$, $z = 1 - t^2$, $t \in [1, 0]$

$x = \cos t$, $y = \sin t$, $z = 0$, $t \in [0, \frac{\pi}{2}]$

$x = t$, $y = 0$, $z = 1 - t^2$, $t \in [0, 1]$

STOKES LHS

$$\oint_C \underline{F} \cdot d\underline{r} = \oint_C zy\,dx + x\,dy + 1\,dz$$

$$W = \overbrace{\int_0^1 0 + 0 + 1(-2t\,dt)}^{C_1} + \underbrace{\int_{t=0}^{\pi/2} (\cos t)(\cos t\,dt)}_{C_2} + \underbrace{\int_1^0 1(-2t\,dt)}_{C_3}$$

$\boxed{6}$

$$W = \cancel{\int_0^1 -2t\,dt} + \int_0^{\pi/2} \cos^2 t\,dt \quad \cancel{\int_1^0 -2t\,dt}$$

$$= \int_0^{\pi/2} \frac{\cos(2t) + 1}{2}\,dt$$

$$= \cancel{\left[\frac{\sin(2t)}{2 \cdot 2}\right]_0^{\pi/2}} + \left[\frac{t}{2}\right]_0^{\pi/2} = \frac{\pi}{4}$$

STOKES RHS

$$W_2 = \iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{n}\, dS$$

$\boxed{7}$

Curl:

$$\underline{\nabla} \times \underline{F} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} zy \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 0-0 \\ y-0 \\ 1-z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 1-z \end{bmatrix}$$

Normal

$$G = z + x^2 + y^2 - 1$$

$$\underline{\nabla} G = \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix}$$

$$\underline{n} = \frac{1}{\sqrt{1+4x^2+4y^2}} \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix}$$

Surf-elem

$$dS = \sqrt{1+4x^2+4y^2}\, dA$$

Region

$\boxed{8}$

$$W_2 = \iint_R \begin{bmatrix} 0 \\ y \\ 1-z \end{bmatrix} \cdot \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix} \cancel{\frac{\sqrt{1+4x^2+4y^2}}{\sqrt{1+4x^2+4y^2}}}\, dA$$

$$z = 1 - x^2 - y^2$$

$$= \iint_R 2y^2 + \left(1 - (1 - x^2 - y^2)\right) dA$$

$$= \iint_R (2y^2 + x^2 + y^2)\, dA$$

Use polar

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} (2r^2 \sin^2\theta + r^2)\, r\, dr\, d\theta$$

$$= \ldots \text{some work} \ldots = \frac{\pi}{4}$$

$\boxed{9}$

$$\cos(2a) = \cos^2(a) - \sin^2(a)$$

$$= \cos^2(a) - \left(1 - \cos^2(a)\right)$$

$$= 2\cos^2(a) - 1$$

$$\cos^2(a) = \frac{\cos(2a) + 1}{2}$$