

Appl. Maths. B242-2023: LECTURE 17

LECTURE 17 [9.12] SURFACE INT'S + FLUX [1]

$$I = \iint_S g(x, y, z) dS$$

$$z = f(x, y)$$

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dA$$

Convert to flat region, double integral in xy-plane.

PROOF:  $dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dA$



$A_{\square} = bh$   
 $= b a \sin \theta$   
 $= ab \sin \theta$   
 $= \|\underline{a} \times \underline{b}\|$



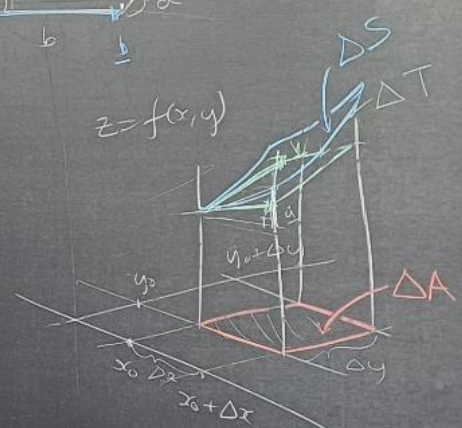
As  $\Delta x \rightarrow 0$ , and  $\Delta y \rightarrow 0$ ,

$\Delta A \rightarrow 0$   
 $\Delta S \rightarrow \Delta T$

$\Delta T = \|\underline{a} \times \underline{b}\|$

$$\underline{a} = \begin{bmatrix} \Delta x \\ 0 \\ \frac{\partial f}{\partial x} \Delta x \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 0 \\ \Delta y \\ \frac{\partial f}{\partial y} \Delta y \end{bmatrix}$$



$$\Delta T = \left\| \begin{bmatrix} \Delta x \\ 0 \\ \Delta y \end{bmatrix} \times \begin{bmatrix} 0 \\ \Delta y \\ \Delta y f_y \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 0 - \Delta x \Delta y f_x \\ 0 - \Delta x \Delta y f_y \\ \Delta x \Delta y - 0 \end{bmatrix} \right\|$$

$$= \Delta x \Delta y \left\| \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix} \right\|$$

$$= \Delta A \sqrt{1 + f_x^2 + f_y^2}$$

$$\Delta S \rightarrow \Delta T \rightarrow \Delta A \sqrt{1 + f_x^2 + f_y^2}$$

Take limit.  $dS = \sqrt{1 + f_x^2 + f_y^2} dA$

$$\|\lambda \mathbf{a}\| = |\lambda| \|\mathbf{a}\|$$

Uses:

① area  $A = \iint_S dS$

② mass  $m = \iint_S \rho(x, y, z) dS$

③ coords of centroid

$$\begin{cases} m\bar{x} = \iint_S x \rho dS \\ m\bar{y} = \iint_S y \rho dS \\ m\bar{z} = \iint_S z \rho dS \end{cases}$$



④ moments of inertia

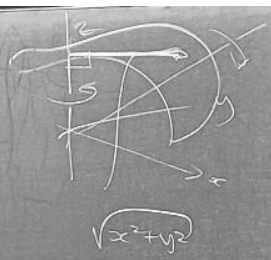
$$I_{zz} = \iint_S \rho \sqrt{x^2 + y^2}^2 dS$$

$I_{xx}$   $I_{yy}$

$$\underline{M} = \underline{I} \underline{\alpha}$$



- ① flux through  $S$ .
- ② flux out of a body  $B$ .

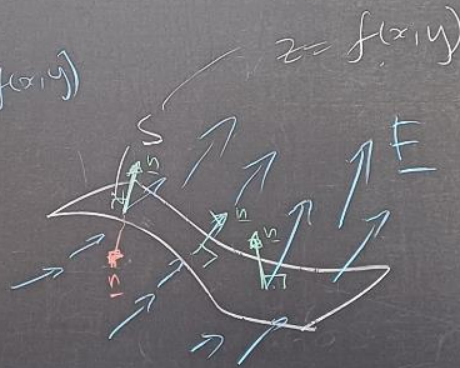


Flux:

upward

↑ flux of  $E$  through  $S$

$$= \iint_S E \cdot n \, dS$$



$n$  is "upwards" if its  $z$ -component is positive.

Example: Find <sup>upward</sup> flux of  $E$  through  $S$ .

$$S: z = 1 - x^2 - y^2, z \geq 0$$

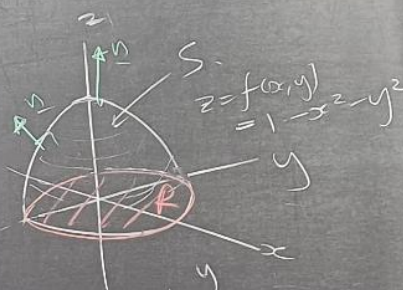
$$E = \begin{bmatrix} y \\ x \\ 3 \end{bmatrix}$$

$S \rightarrow R$

Normal

$$G = 1 - x^2 - y^2 - z$$

$$-\nabla G = \begin{bmatrix} +2x \\ +2y \\ +1 \end{bmatrix} \frac{1}{\sqrt{1 + (2x)^2 + (2y)^2}}$$



$$n = \frac{1}{\sqrt{1 + 4x^2 + 4y^2}} \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix}$$

Surface element:

$$z = 1 - x^2 - y^2$$

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$$dS = \sqrt{1 + (-2x)^2 + (-2y)^2} dA$$
$$= \sqrt{1 + 4x^2 + 4y^2} dA$$

$$\text{flux} = \iint_S \mathbf{F} \cdot \mathbf{n} dS$$

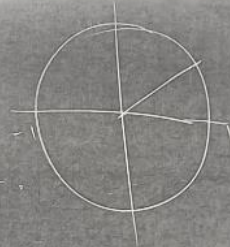
$$= \iint_R \begin{bmatrix} y \\ x \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix} \frac{1}{\sqrt{1+4x^2+4y^2}} \sqrt{1+4x^2+4y^2} dA$$

$$= \iint_R (4xy + 3) dA$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^1 (4r \cos \theta r \sin \theta + 3) r dr d\theta$$

⋮  
work  
⋮

$$= 3\pi$$



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