Appl.Maths. B242-2023: **LECTURE 16**

LECTURE 16 — [9.13] SURFACE INTEGRALS

1

Double integral:

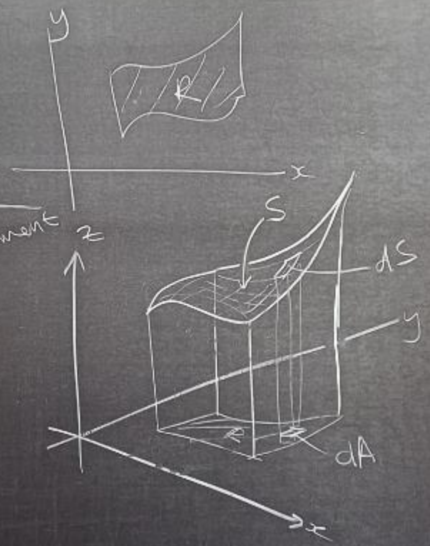
$$I = \iint_R f(x,y)\,dA$$

Surface integral:

$$I = \iint_S g(x,y,z)\,\underbrace{dS}_{\text{area-element}}$$

Surf. int's:

- $g(x,y,z)$
- $S:\ z = f(x,y)$
- $(x,y) \in R$



2

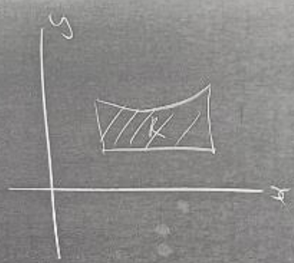
Convert surf-int to double int

$$I = \iint_S g(x,y,z)\,dS$$

S: Proj. S on R

$z = f(x,y)$

$$dS = \sqrt{1 + (f_x)^2 + (f_y)^2}\;dA$$



$$I = \iint_R g(x,y,f(x,y))\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\;dA$$

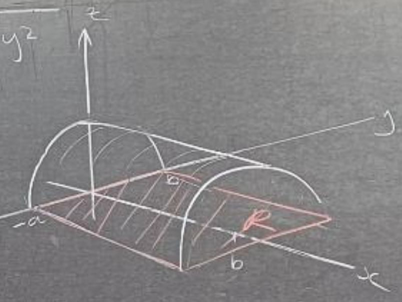
$$f_x = \frac{\partial f}{\partial x}$$

$$\iint_S 1\,dS = \text{area}$$

$\boxed{3}$

Example: a and b are positive constants.

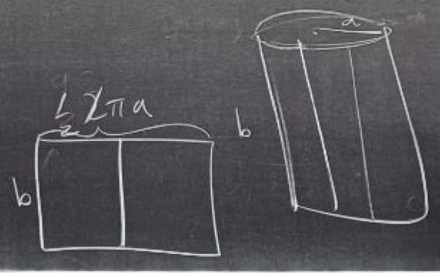
$$S: \begin{cases} z^2 + y^2 = a^2, \quad z = \pm\sqrt{a^2 - y^2} \\ x \in [0, b] \\ z \geq 0 \end{cases}$$



Calculate area of S.

$$\text{Area} = \iint_S 1 \, dS$$

$$\text{Area} = \frac{1}{2}(2\pi a b) = \pi a b$$

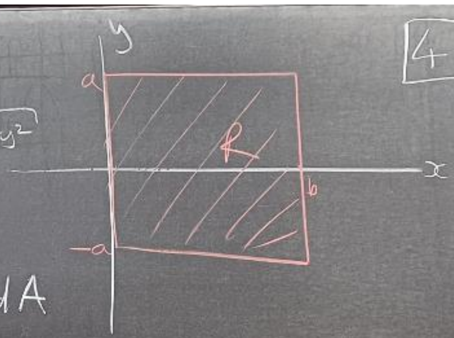


$$\frac{1}{2} \cdot 2\pi a$$

$\boxed{4}$

$$\text{Area} = \iint_{\underset{R}{\downarrow}{S}} 1 \, dS \qquad z = \sqrt{a^2 - y^2}$$

$$f(x,y) = \sqrt{a^2 - y^2}$$



$$dS = \sqrt{1 + (0)^2 + \left(\frac{-\cancel{2}y}{\cancel{2}\sqrt{a^2 - y^2}}\right)^2} \, dA$$

$$= \sqrt{1 + \frac{y^2}{a^2 - y^2}} \, dA$$

$$= \sqrt{\frac{a^2 - \cancel{y^2} + \cancel{y^2}}{a^2 - y^2}} \, dA = \frac{a}{\sqrt{a^2 - y^2}} \, dA$$

$\boxed{5}$

$$I = \iint_R 1 \frac{a}{\sqrt{a^2-y^2}}\, dA = \int_{x=0}^{b}\left[\int_{y=-a}^{a} \frac{a}{\sqrt{a^2-y^2}}\, dy\right] dx$$

$$= \left[\int_{x=0}^{b} 1\, dx\right] \times \left[\int_{y=-a}^{a} \frac{a}{\sqrt{a^2-y^2}}\, dy\right]$$

$$= b\, a \int_{-a}^{a} \frac{1}{\sqrt{a^2-y^2}}\, dy$$

Subst: $y = a \sin u$

$dy = a \cos u \, du$.

$y = -a \longrightarrow u = -\pi/2$

$y = a \longrightarrow u = \pi/2$

$$= ab \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{a^2 - a^2 \sin^2 u}}\, a \cos u \, du$$

$$= a^{\cancel{2}} b \int_{-\pi/2}^{\pi/2} \frac{\cancel{\cos u}}{a \cancel{\cos u}}\, du = ab \Big[u\Big]_{-\pi/2}^{\pi/2} = ab\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

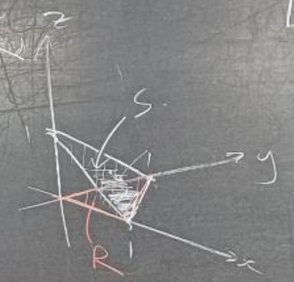
$$= \pi ab$$

$\boxed{6}$

Example 2. ~~Find mass~~

$S: x+y+z=1$, only positive octant

Density: $\rho = kx^2$



$$\text{mass} = \iint_S \rho \, dS$$

$$= \iint_S kx^2 \, dS \quad \leftarrow \sqrt{3}\, dA$$

$$= \iint_R kx^2 \sqrt{3}\, dA$$

$$m = \sqrt{3}\, k \int_{x=0}^{1}\int_{y=0}^{1-x} x^2 \, dy\, dx = \frac{\sqrt{3}\, k}{12}$$

$S: z = 1 - x - y$

$$dS = \sqrt{1 + (-1)^2 + (-1)^2}\, dA$$

$$= \sqrt{3}\, dA$$