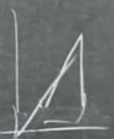
Appl.Maths. B242-2023: LECTURE 15

LECTURE 15 [9.12] GREEN'S THEOREM

$\boxed{1}$

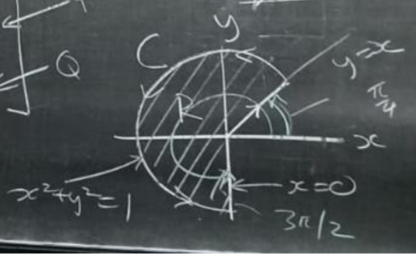
$$\oint_C \underline{F}\cdot d\underline{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\underline{F}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$



Example: Use Green to calculate

$$W = \oint_C \underline{F}\cdot d\underline{r}, \quad \underline{F} = \begin{array}{cc} \begin{bmatrix} (x-x^2)y \\ xy^2 \end{bmatrix} & \begin{array}{l} \leftarrow P \\ \leftarrow Q \end{array} \end{array}$$



$\boxed{2}$

"Use"

$$W = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}\left(xy^2\right) - \frac{\partial}{\partial y}\left(xy - x^2y\right)$$

$$= y^2 - \left(x - x^2\right)$$

$$= x^2 + y^2 - x$$

$$= \iint_R \left(x^2+y^2-x\right) dA$$

Use Polar

$$= \int_{\theta=\pi/4}^{3\pi/2} \int_{r=0}^{1} \left(r^2 - r\cos\theta\right) r\,dr\,d\theta$$

$$= \int_{\theta=\pi/4}^{3\pi/2} \int_{r=0}^{1} \left(r^3 - r^2\cos\theta\right) dr \;\; d\theta$$

$$N = \int_{\pi/4}^{3\pi/2} \left[\frac{r^4}{4} - \frac{r^3}{3}\cos\theta\right]_{r=0}^{1} d\theta = \int_{\pi/4}^{3\pi/2} \left(\frac{1}{4} - \frac{1}{3}\cos\theta\right) d\theta$$

$$= \left[\frac{1}{4}\theta\right]_{\pi/4}^{3\pi/2} - \frac{1}{3}\left[\sin\theta\right]_{\pi/4}^{3\pi/2}$$

$$= \frac{1}{4}\left(\frac{6\pi}{4} - \frac{\pi}{4}\right) - \frac{1}{3}\left[-1 - \frac{1}{\sqrt{2}}\right]$$

$$= \frac{5\pi}{16} + \frac{\sqrt{2}+1}{3\sqrt{2}} = 1.551$$

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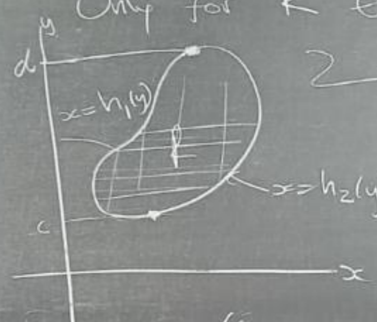
PARTIAL PROOF OF GREEN

Only for R that is Type 1 or Type 2.



$$RHS = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_R \frac{\partial Q}{\partial x}\,(dx\,dy) - \iint_R \frac{\partial P}{\partial y}\,(dy\,dx)$$

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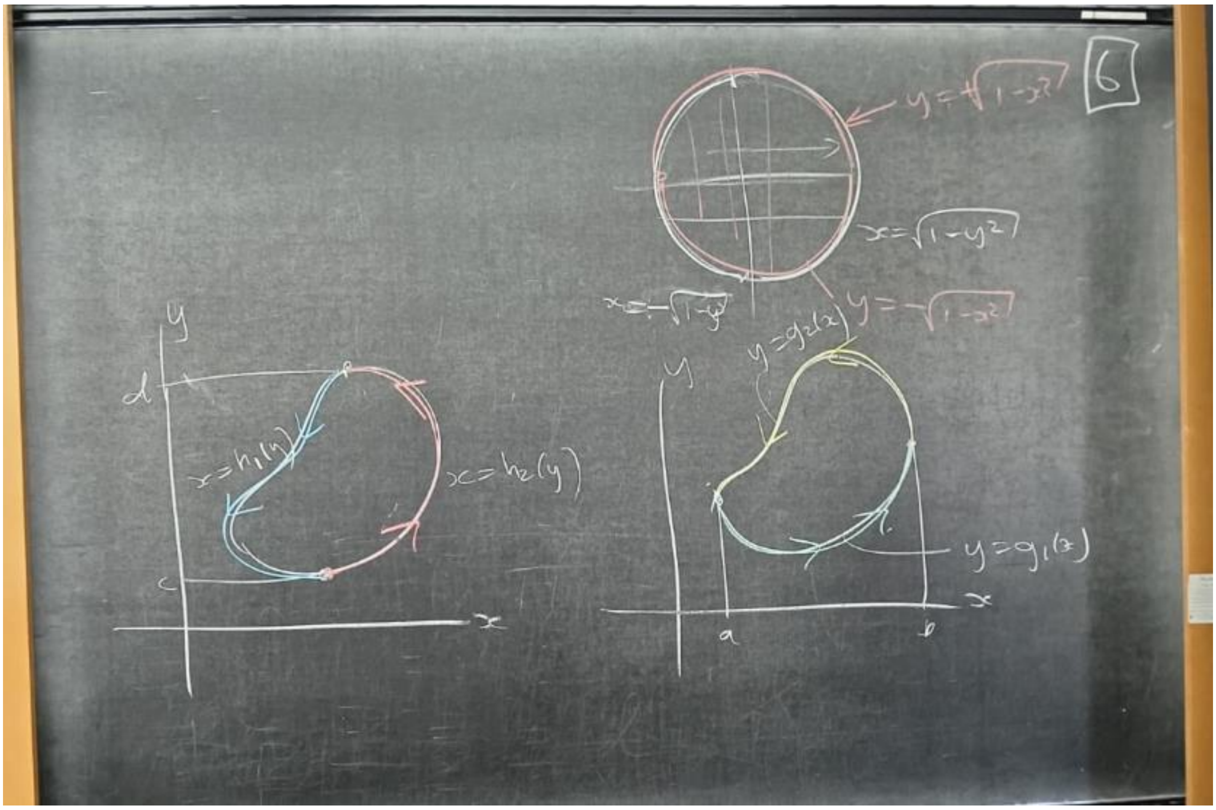
$$= \int_{y=c}^{d}\left[\int_{x=h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial x}\,dx\right]dy - \int_{x=a}^{b}\left[\int_{y=g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\right]dx$$

$$= \int_{y=c}^{d}\Big(Q(h_2(y),y) - Q(h_1(y),y)\Big)\,dy$$

$$\qquad - \int_{x=a}^{b}\Big(P(x,g_1(x)) - P(x,g_2(x))\Big)\,dx$$

$$= \underline{\int_{y=c}^{d} Q(h_2(y),y)\,dy} + \int_{d}^{c} \underline{Q(h_1(y),y)\,dy}$$

$$+ \underline{\int_{b}^{a} P(x,g_2(x))\,dx} + \int_{a}^{b} \underline{P(x,g_1(x))\,dx}$$



$\boxed{7}$

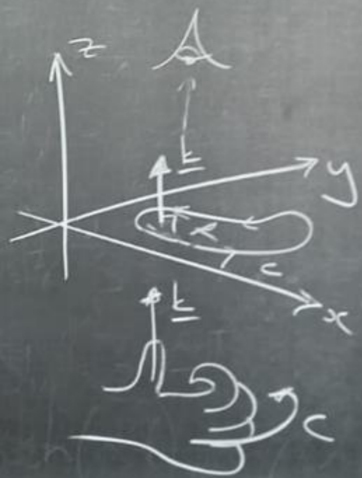
$$= \oint_C Q\,dy + \oint_C P\,dx = \text{LHS}.$$

Summary: Green in 3D

$$\underline{F}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \\ 0 \end{bmatrix}$$

$$\oint_C \underline{F}\cdot d\underline{r} = \oint P\,dx + Q\,dy + \cancel{0\,dz}$$

$$= \iint_R (\underline{\nabla}\times\underline{F})\cdot\underline{k}\,dA$$



$\boxed{8}$

$$\underline{\nabla}\times\underline{F} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} P(x,y) \\ Q(x,y) \\ 0 \end{bmatrix} = \begin{bmatrix} 0-0 \\ 0-0 \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{bmatrix}$$

$$\underline{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{e}_x = \underline{i}$$

STOKES:

$$\oint_C \underline{F}\cdot d\underline{r} = \iint_S \underbrace{(\underline{\nabla}\times\underline{F})}_{\checkmark}\cdot\underbrace{\underline{n}}_{\checkmark}\,\underbrace{dS}_{\text{must be explained}}$$