Appl.Maths. B242-2023: LECTURE 14

LECTURE 14 [9.11] POLAR, [9.12] GREEN

1

Off centre circles:

Cartesian

$$x^2+(y-2)^2=3^2$$

$$x^2+y^2-4y+4=9$$

$$\boxed{x^2+y^2-4y=5}$$

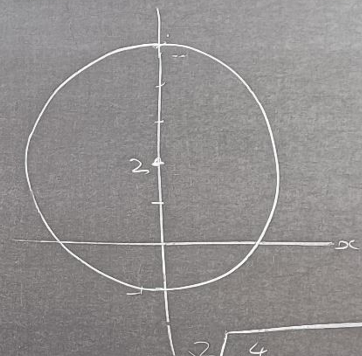
$\boxed{r^2=x^2+y^2}$

$$\boxed{\begin{aligned} x&=r\cos\theta\\ y&=r\sin\theta\end{aligned}}$$

Polar

$$r^2-4r\sin\theta=5$$

$$r^2-4\sin\theta\, r-5=0$$



$$r=\frac{4\sin\theta\pm\overset{2}{}\sqrt{\overset{4}{\cancel{16}}\sin^2\theta+\overset{5}{\cancel{20}}}}{2}$$

$$r=2\sin\theta\pm\sqrt{4\sin^2\theta+5}$$

2

Example:

Find area of R.
(use polar)

$$\text{Area}=\iint_R 1\,dA$$

$$=\int_{\theta=0}^{\pi}\int_{r=0}^{2\cos\theta}1\;r\,dr\,d\theta$$

$$=\pi$$

also

$$=\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r=0}^{2\cos\theta}1\;r\,dr\,d\theta=\pi$$

$$(x-1)^2+y^2=1$$

$$x^2-2x+1+y^2=1$$

$$x^2+y^2-2x=0$$

$\longrightarrow$ to polar

$$r^2-2r\cos\theta=0$$

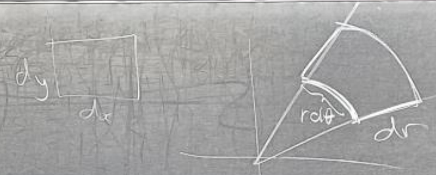
$$r(r-2\cos\theta)=0$$

$r=0$ or $\boxed{r=2\cos\theta}$

3

$dA = dx\,dy$

$dA = (r\,d\theta)\,dr$



Theorem:

— Green's (flat surf in 2D)

— Stokes' (curved surf in 3D)

— Gauss [Divergence] (volume f. in 3D)

4

GREEN'S TH

$$\underline{F}(x,y) = \underline{i}\,P(x,y) + \underline{j}\,Q(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

$$\boxed{\underbrace{\oint_C \underline{F}\cdot d\underline{r}}_{\text{circulation of } \underline{F}} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Example:

$$\underline{F} = \begin{bmatrix} x - y \\ xy \end{bmatrix}$$

C_3: $x = t$, $y = 3t$, $t \in [1,0]$

C_2: $x = 1$, $y = t$, $t \in [0,3]$

C_1: $x = t$, $y = 0$, $t \in [0,1]$

C is closed, and simple

$\boxed{5}$

LHS: $\oint_C \underline{F}\cdot d\underline{r}$

$$= \int_C (x-y)\,dx + xy\,dy = \int_0^1 (t-0)\,dt + \int_0^3 \cancel{(1-t)} + (1t)\,dt$$

$$+ \int_1^0 (t-3t)\,dt + (t(3t))(3dt)$$

$(9t^2-2t)$

$$= \left[\frac{t^2}{2}\right]_0^1 + \left[\frac{t^2}{2}\right]_0^3 + \left[\frac{9t^3}{3} - \frac{2t^2}{2}\right]_1^0$$

$$= \frac{1}{2} + \frac{9}{2} + \left(0 - (3-1)\right) = \underline{\underline{3}}$$

$\boxed{6}$

$$\text{RHS} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x-y)$$

$$= y - (-1) = y+1$$

$$= \iint_R (y+1)\,dA$$

<u>Use type 1</u>:

$$\text{RHS} = \int_{x=0}^{1} \left(\int_{y=0}^{3x} (y+1)\,dy \right) dx$$

$$= \int_0^1 \left[\frac{y^2}{2} + y\right]_0^{3x} dx$$

$$= \int_0^1 \left(\frac{9x^2}{2} + 3x\right) dx = \left[\frac{9}{2}\cdot\frac{x^3}{3} + \frac{3x^2}{2}\right]_0^1 = \underline{\underline{3}}$$