Appl.Maths. B242-2023: LECTURE 11

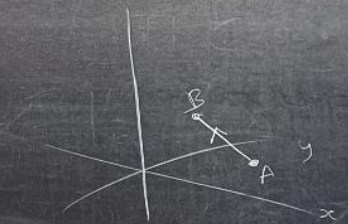
LECTURE 11 [9.10] DOUBLE INTEGRALS (1)

Last example: Path int's + Path Independence in 3D

$$\underline{F} = \begin{bmatrix} y + yz^2 + 1 \\ x + xz^2 + 2z \\ 2xyz + 2y \end{bmatrix} \qquad W = \int_{(1,4,0)}^{(2,2,1)} \underline{F}\cdot d\underline{r}$$

Do both ways: with path / without path.

With path:



$$\underline{r} = \underline{r}_1 + t\underline{a} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \left.\begin{aligned} x &= 1+t \\ y &= 4-2t \\ z &= t \end{aligned}\right\} t \in [0,1]$$

(2)

$$W = \int (y+yz^2+1)\,dx + (x+xz^2+2z)\,dy + (2xyz+2y)\,dz$$

$$= \int_0^1 \left[(4-2t)+(4-2t)t^2+1\right]dt + \left(1+t+(1+t)t^2+2t\right)(-2\,dt) + \ldots\ldots$$

some work

$$= \ldots\ldots \quad 9$$

Without path:

Test:
$$\nabla\times\underline{F} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} y + yz^2 + 1 \\ x + xz^2 + 2z \\ 2xyz + 2y \end{bmatrix} = \begin{bmatrix} (2xz+2) - (2xz+2) \\ 2zy - 2zy \\ (1+z^2) - (1+z^2) \end{bmatrix} = \underline{0}$$

$$\nabla\times\underline{F} = \underline{0}$$

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$$\boxed{\nabla \times (\nabla \varphi) = \underline{0}}$$

Find φ: (Term collection)

$$\varphi = \int P\,dx = \int (y + yz^2 + 1)\,dx = yx + xyz^2 + x + \ldots\ldots$$

$$\varphi = \int Q\,dy = \int (x + xz^2 + 2z)\,dy = xy + yxz^2 + 2yz + \ldots\ldots$$

$$\varphi = \int R\,dz = \int (2xyz + 2y)\,dz = 2xy\frac{z^2}{2} + 2yz + \ldots\ldots$$

$$\varphi = xyz^2 + xy + x + 2yz + C$$

Check: $\frac{\partial \varphi}{\partial x} = \ldots\ldots P \qquad \frac{\partial \varphi}{\partial y} \ldots\ldots Q$

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$$W = \int_{\bullet A}^{\bullet B} \underline{F} \cdot d\underline{r} = \int_{\bullet A}^{\bullet B} d\varphi = \Big[\varphi(x,y,z)\Big]_{\bullet A}^{\bullet B}$$

$$= \Big[\varphi\Big]_{(1,4,0)}^{(2,2,1)} = \Big[xyz^2 + xy + x + 2yz\Big]_{(1,4,0)}^{(2,2,1)}$$

$$= (2\cdot 2\cdot 1 + 2\cdot 2 + 2 + 2\cdot 2\cdot 1) - (0 + 1\cdot 4 + 1 + 0)$$

$$= 14 - 5 = 9$$

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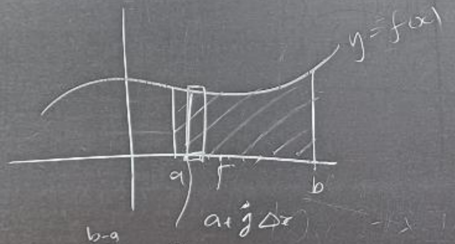

$$\int_a^b f(x)\,dx = \text{area}$$

$$\text{area} \approx \sum_{j=0}^{\frac{b-a}{\Delta x}-1} f(x_j)\,\Delta x$$

$$\Delta x = \frac{b-a}{N}$$

$$N = \frac{b-a}{\Delta x}$$

$$\lim_{\Delta x \to 0}$$

$$\text{area} = \int_a^b f(x)\,dx$$

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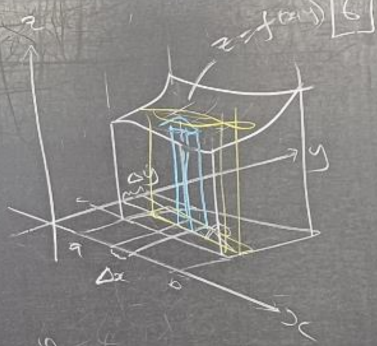
Double int

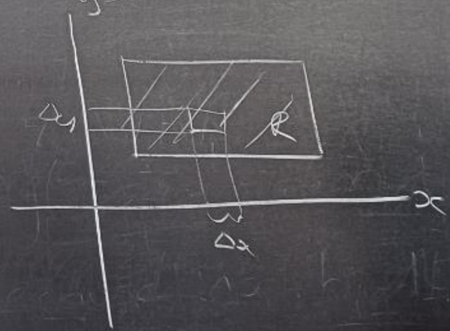
$$z = f(x,y)$$



$$\sum_{k=0}^{M-1} \sum_{j=0}^{N-1} f(x_j, y_k)\,\Delta x\,\Delta y \approx \text{volume}$$

$$\text{volume} = \iint_R f(x,y)\,dx\,dy$$

$$= \iint_R f(x,y)\,dy\,dx$$



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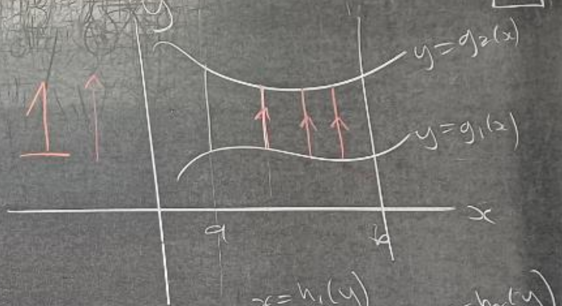
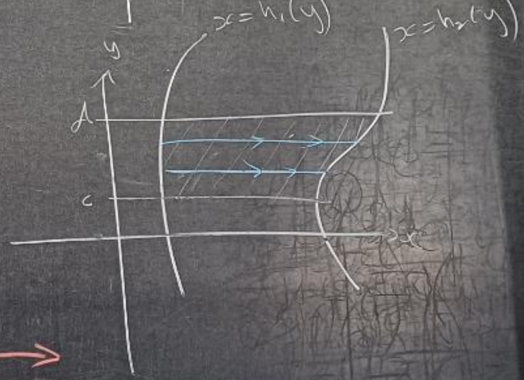
Type 1 region:

$$\int_{x=a}^{b}\left[\int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\right]dx \qquad 1\uparrow$$



Type 2 region:

$$\int_{y=c}^{d}\left[\int_{x=h_1(y)}^{h_2(y)} f(x,y)\,dx\right]dy \qquad 2\rightarrow$$



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Example: $I = \iint_R 2xy\,dA$ $\qquad dA = dx\,dy$

R: region bounded by $\left.\begin{array}{l} y = x^3 \\ y = 8 \\ x = 0 \end{array}\right\}$

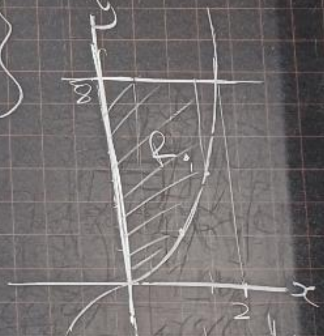
As type 1: 1

$$\int_{x=0}^{2}\left[\int_{y=x^3}^{8} 2xy\,dy\right]dx$$

$y \geq x^3$

$y \leq 8$

$x \geq 0$



$$= \int_0^2 \left[2x\frac{y^2}{2}\right]_{x^3}^{8} dx$$

$$= \int_0^2 \left[xy^2\right]_{y=x^3}^{8} dx = \int_0^2 \left[64x - x(x^3)^2\right] dx$$

$\boxed{9}$

$$= \int_0^2 (64x - x^7)\,dx = \left[\frac{64x^2}{2} - \frac{x^8}{8}\right]_0^2 = 96.$$

As type 2: $\longrightarrow$

$$\int_{y=0}^{8} \left[\int_{x=0}^{y^{1/3}} 2xy \, dx \right] dy$$