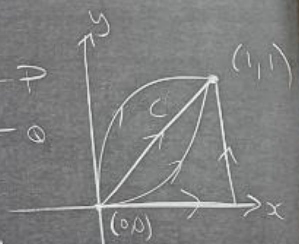
Appl.Maths. B242-2023: LECTURE 10

LECTURE 10 [9.9] PATH INDEPENDENCE

1

Example: $W = \int_C \underline{F}\cdot d\underline{r}$, $\underline{F} = \begin{bmatrix} y \\ x \end{bmatrix} \begin{matrix} \leftarrow P \\ \leftarrow Q \end{matrix}$



- $\underline{F}$ is a conservative field.
- $\underline{F}\cdot d\underline{r}$ is an exact differential, $\boxed{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} = 0 \checkmark}$
- $\varphi(x,y)$ exists such that $\underline{F}\cdot d\underline{r} = d\varphi$
- How to integrate using φ.

$$\underline{F}\cdot d\underline{r} = \begin{bmatrix} P \\ Q \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix} = P\,dx + Q\,dy$$

2

Chain rule:

$\varphi(x,y) \qquad \varphi(x(s), y(s))$

$$\frac{d\varphi}{\cancel{ds}} = \frac{\partial \varphi}{\partial x}\frac{dx}{\cancel{ds}} + \frac{\partial \varphi}{\partial y}\frac{dy}{\cancel{ds}}$$

$$d\varphi = \varphi_x\,dx + \varphi_y\,dy$$

$$\| $$

$$\underline{F}\cdot d\underline{r} = P\,dx + Q\,dy$$

$$\frac{\partial}{\partial y}\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial y}P \qquad \Bigg| \qquad \frac{\partial}{\partial x}\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial x}Q$$

- Test for E.D.

$$\varphi_{xy} = \frac{\partial P}{\partial y} \qquad \qquad \varphi_{yx} = \frac{\partial Q}{\partial x}$$

3

same if φ is ~~smooth~~ smooth enough

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

If $\boxed{\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0}$, then $\underline{F}$ is conservative.

Find φ:

$$\frac{\partial \varphi}{\partial x} = P$$

$$\int d\varphi = \int P\,dx$$

$$\varphi = \ldots x, y \ldots + f(y)$$

$$\frac{\partial \varphi}{\partial y} = Q$$

$$d\varphi = Q\,dy$$

$$\varphi = \ldots x, y \ldots + g(x)$$

4

Example:

$$\underline{F} = \begin{array}{c} \begin{bmatrix} y^2 \; + 1 \\ 2y\ln x + 3y^2 \end{bmatrix} \end{array} \begin{array}{l} P \\ Q \end{array}$$

Is $\underline{F}$ conservative?, Find φ.

Test

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{2y}{x} - \frac{2y}{x} = 0$$

$d\varphi$ is an ED

Find φ:

$$\varphi = \int P\,dx$$

$$\varphi = \int \left(\frac{y^2}{x} + 1\right) dx$$

$$= \underline{\underline{y^2 \ln x}} + x + \underbrace{\ldots\ldots}_{\text{depend on } y}$$

$$\varphi = \int Q\,dy$$

$$\varphi = \int \left(2y\ln x + 3y^2\right) dy$$

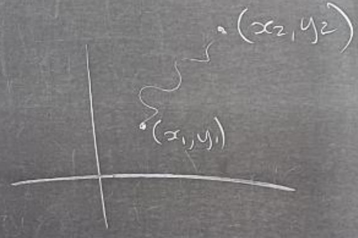
$$= \underline{\underline{2\frac{y^2}{2}\ln x}} + \frac{3y^3}{3} + \underbrace{\ldots\ldots}_{\text{depend on } x}$$

Compare:

$$\boxed{\phi = y^2 \ln x + x + y^3 + C}$$

5

① <u>Integrate with ϕ:</u>



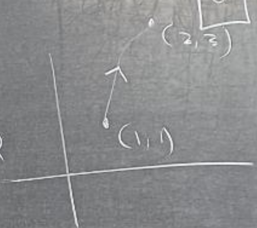
$$W = \int_C \underline{F} \cdot d\underline{r} = \int d\phi$$

$$= \Big[\phi(x,y)\Big]_{(x_1,y_1)}^{(x_2,y_2)}$$

<u>Example (both ways)</u>

6

$$W = \int_C \underline{F} \cdot d\underline{r}, \quad \underline{F} = \begin{bmatrix} y + 6x^2 \\ x + 2y \end{bmatrix} \begin{matrix} \leftarrow P \\ \leftarrow Q \end{matrix}$$



<u>With path:</u>

$$W = \int_C (y + 6x^2)\,dx + (x + 2y)\,dy$$

$$= \int_0^1 \Big[(\overset{1}{1} + \overset{2}{2t}) + \overset{6}{6}(1+t)^{\overset{12}{2}}\Big] dt$$

$$+ \Big(\overset{2}{1} + \underset{2}{t} + 2(\overset{4}{1} + \underset{8}{2t})\Big)(2\,dt)$$

$$= \int_0^1 (13 + 24t + 6t^2)\,dt = 27$$

$$\underline{r} = \underline{r}_1 + t\,\underline{a}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

C:

$$\left.\begin{matrix} x = 1 + t \\ y = 1 + 2t \end{matrix}\right\} \quad t \in [0,1]$$

7

With ϕ

Test: $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1 - 1 = 0$ ✓

Find ϕ:

$$\phi = \int P \, dx \qquad\qquad \phi = \int Q \, dy$$

$$\phi = \int (y + 6x^2) \, dx \qquad\qquad \phi = \int (x + 2y) \, dy$$

$$= xy + \frac{6x^3}{3} + \cdots \qquad\qquad \phi = xy + \frac{2y^2}{2} + \cdots$$

$$\phi = xy + 2x^3 + y^2 + C$$

8

Integrate:

$$W = \int_{(1,1)}^{(2,3)} \underline{F} \cdot d\underline{r} = \int_{(1,1)}^{(2,3)} d\phi = \Big[\phi(x,y)\Big]_{(1,1)}^{(2,3)}$$

$$= \Big[xy + 2x^3 + y^2\Big]_{(1,1)}^{(2,3)}$$

$$= \left(2 \cdot 3 + 2(2)^3 + 3^2\right) - \left(1 \cdot 1 + 2 \cdot 1 + 1\right)$$

$$= 6 + 16 + 9 - 4 = 27$$

F is conservative

$\equiv$ F is a gradient field

$\equiv$ $\underline{F} \cdot d\underline{r}$ is an exact differential

$\equiv$ ϕ exists so that $\underline{F} \cdot d\underline{r} = d\phi$

$\boxed{9}$

$$\equiv \int_A^B \underline{F}\cdot d\underline{r} = \int_A^B d\varphi = \Big[\varphi\Big]_A^B$$

3D $\quad \underline{F}(x,y,z) = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}, \quad W = \int \underline{F}\cdot d\underline{r}$

Find φ:

$\varphi(x,y,z)$

$$d\varphi = \varphi_x\,dx + \varphi_y\,dy + \varphi_z\,dz$$

$$\parallel$$

$$\underline{F}\cdot d\underline{r} = P\,dx + Q\,dy + R\,dz$$

$\boxed{10}$

$$\underline{F} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{bmatrix} = \nabla\varphi$$

$\varphi_{xz} = \varphi_{zx}$

$$\varphi_x = P, \quad \varphi_y = Q, \quad \varphi_z = R$$

$\varphi_{xy} = \varphi_{yx} \qquad \varphi_{yz} = \varphi_{zy}$

Test

$$\varphi_{xy} = P_y = \varphi_{yx} = Q_x$$

$$\varphi_{yz} = Q_z = \varphi_{zy} = R_y$$

$$P_z = R_x$$

$$\underline{\nabla} \times \underline{F} = \underline{0}$$

$P_y = Q_x$

$Q_z = R_y$

$P_z = R_x$

$$\begin{bmatrix} R_y - Q_z = 0 \\ P_z - R_x = 0 \\ Q_x - P_y = 0 \end{bmatrix}$$

$$\downarrow$$

$$\underline{\nabla} \times \underline{F} = \underline{0}$$

$\varphi_x = P \;\Big|\; \varphi_y = Q, \;\Big|\; \varphi_z = R$