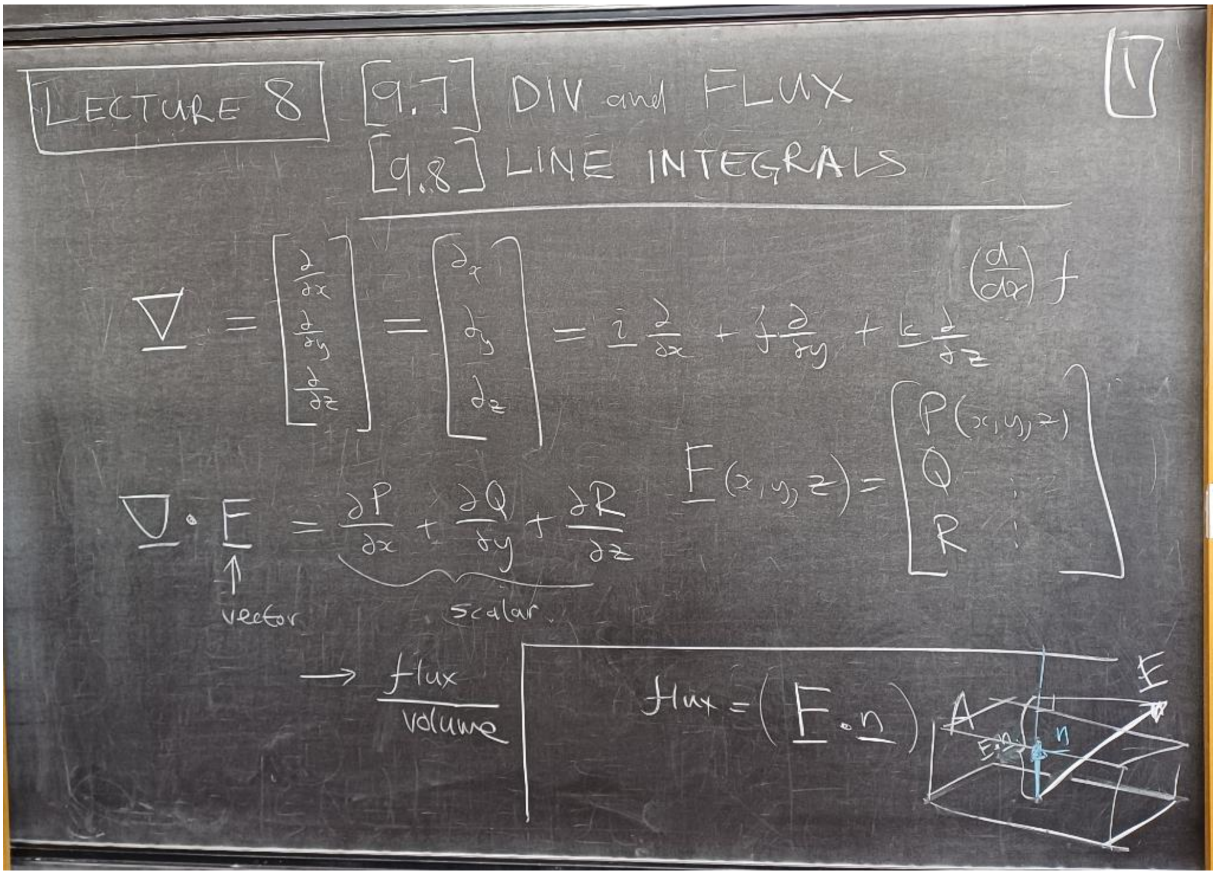
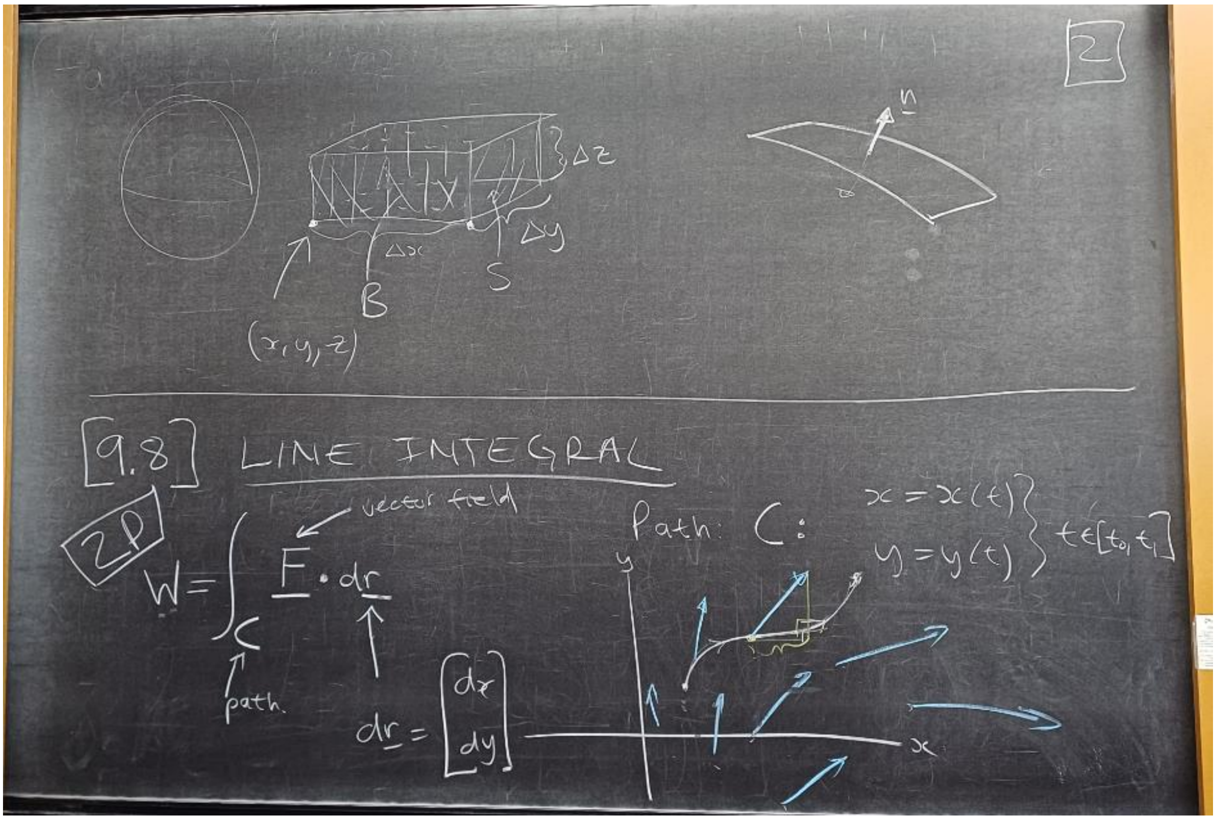
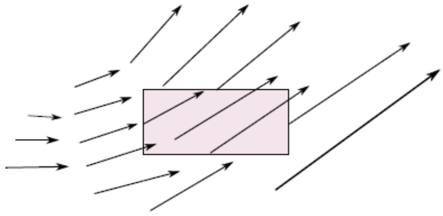
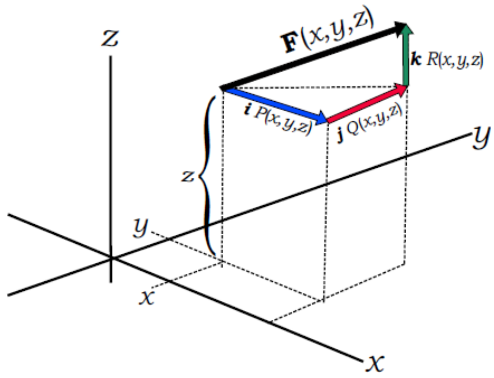
Appl.Maths. B242-2023: LECTURE 8





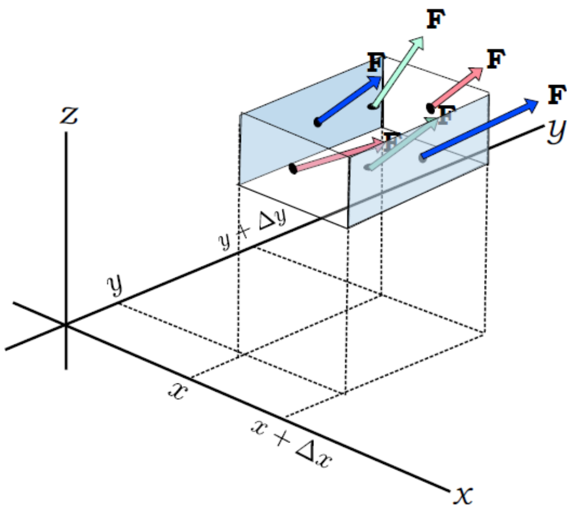
Geometrical interpretation of DIVERGENCE:



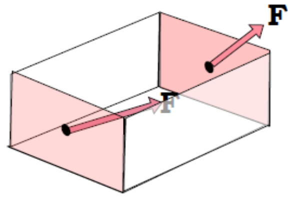


In 3D:

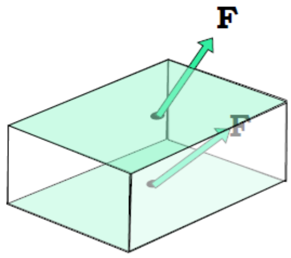
$$\mathbf{F} = \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix}$$



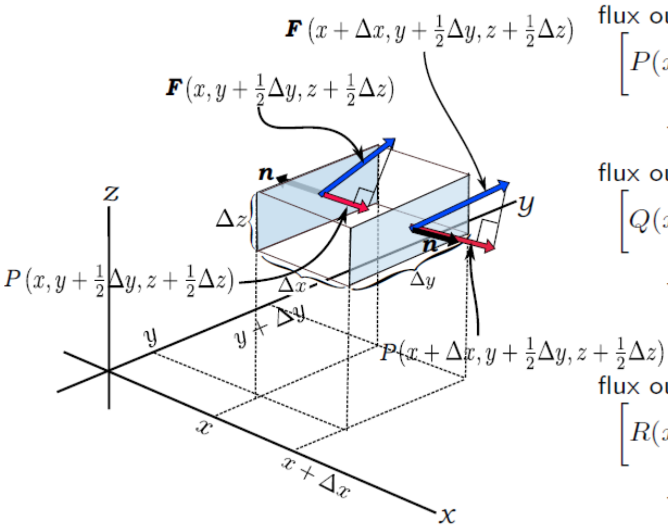
- left and right surfaces are 'blue'



- front and rear surfaces are 'pink'



- top and bottom surfaces are 'green'



flux out of B through blue surfaces =

$$\left[P(x+\Delta x, y+\tfrac{1}{2}\Delta y, z+\tfrac{1}{2}\Delta z) - P(x, y+\tfrac{1}{2}\Delta y, z+\tfrac{1}{2}\Delta z)\right] \Delta y\, \Delta z$$

flux out of B through pink surfaces =

$$\left[Q(x+\tfrac{1}{2}\Delta x, y+\Delta y, z+\tfrac{1}{2}\Delta z) - Q(x+\tfrac{1}{2}\Delta x, y, z+\tfrac{1}{2}\Delta z)\right] \Delta x\, \Delta z$$

flux out of B through green surfaces =

$$\left[R(x+\tfrac{1}{2}\Delta x, y+\tfrac{1}{2}\Delta y, z+\Delta z) - R(x+\tfrac{1}{2}\Delta x, y+\tfrac{1}{2}\Delta y, z)\right] \Delta x\, \Delta y$$

$$\left(\frac{\text{total flux out of B}}{\text{volume of B}}\right)$$

$$= \left[P(x+\Delta x, y+\tfrac{1}{2}\Delta y, z+\tfrac{1}{2}\Delta z) - P(x, y+\tfrac{1}{2}\Delta y, z+\tfrac{1}{2}\Delta z)\right] \frac{\Delta y\, \Delta z}{\Delta x\, \Delta y\, \Delta z}$$
$$+ \left[Q(x+\tfrac{1}{2}\Delta x, y+\Delta y, z+\tfrac{1}{2}\Delta z) - Q(x+\tfrac{1}{2}\Delta x, y, z+\tfrac{1}{2}\Delta z)\right] \frac{\Delta x\, \Delta z}{\Delta x\, \Delta y\, \Delta z}$$
$$+ \left[R(x+\tfrac{1}{2}\Delta x, y+\tfrac{1}{2}\Delta y, z+\Delta z) - R(x+\tfrac{1}{2}\Delta x, y+\tfrac{1}{2}\Delta y, z)\right] \frac{\Delta x\, \Delta y}{\Delta x\, \Delta y\, \Delta z}$$

Take the limit as the volume shrinks to zero:

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \left(\frac{\text{total flux out of B}}{\text{volume of B}}\right)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \left(\left[P(x+\Delta x, y+\tfrac{1}{2}\Delta y, z+\tfrac{1}{2}\Delta z) - P(x, y+\tfrac{1}{2}\Delta y, z+\tfrac{1}{2}\Delta z)\right] / \Delta x\right)$$
$$+ \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \left(\left[Q(x+\tfrac{1}{2}\Delta x, y+\Delta y, z+\tfrac{1}{2}\Delta z) - Q(x+\tfrac{1}{2}\Delta x, y, z+\tfrac{1}{2}\Delta z)\right] / \Delta y\right)$$
$$+ \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \left(\left[R(x+\tfrac{1}{2}\Delta x, y+\tfrac{1}{2}\Delta y, z+\Delta z) - R(x+\tfrac{1}{2}\Delta x, y+\tfrac{1}{2}\Delta y, z)\right] / \Delta z\right)$$

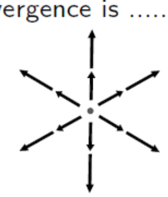
$$\lim_{\text{volume}\to 0}\left(\frac{\text{total flux out of B}}{\text{volume of B}}\right)$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

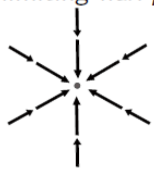
$$= \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \nabla \cdot \mathbf{F}$$

Divergence is ..... *"the flux out of a small body divided by the volume of the body, as this volume tends to zero."*

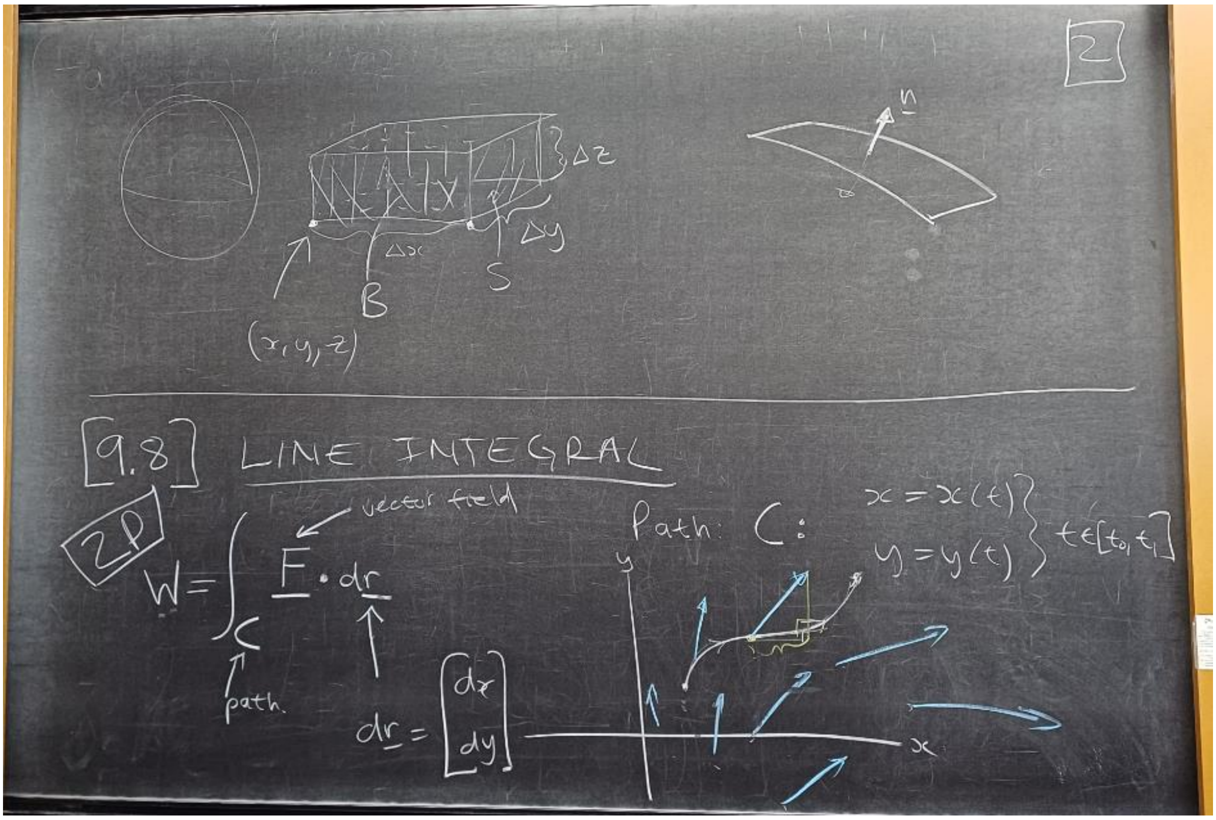
Divergence is ...... *"the limiting flux per volume out of a point."*



Divergence is POSITIVE at this point.



Divergence is NEGATIVE at this point.



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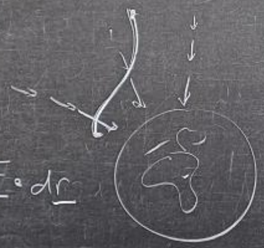
$\underline{r}(t)$

tangent vector: $\underline{r}'(t) = \dfrac{d\underline{r}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} dx \\ dy \end{bmatrix} \dfrac{1}{dt}$

$d\underline{r} = \begin{bmatrix} dx \\ dy \end{bmatrix}$



$$W = \int_C \underline{F} \cdot d\underline{r}$$

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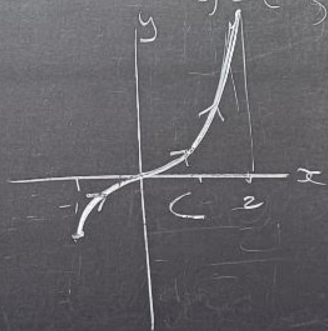
Example: Calc. W

$$W = \int_C \underline{F} \cdot d\underline{r}, \quad \underline{F} = \begin{bmatrix} xy \\ x^2 \end{bmatrix}, \quad \text{~~C: } y = x^3\text{~~}, \; x \in [-1, 2]$$

$$C: \left.\begin{matrix} x = t \\ y = t^3 \end{matrix}\right\} \; t \in [-1, 2]$$



$$W = \int_C \begin{bmatrix} xy \\ x^2 \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$= \int_C xy\,dx + x^2\,dy$$

$$\begin{cases} x = t \\ y = t^3 \\ dx = 1\,dt \\ dy = 3t^2\,dt \end{cases}$$

V

$$W = \int_{-1}^{2} t\,t^3\,dt + t^2\left(3t^2\,dt\right)$$

$$= \int_{-1}^{2} 4t^4\,dt = \left[\frac{4t^5}{5}\right]_{-1}^{2} = 26.4$$

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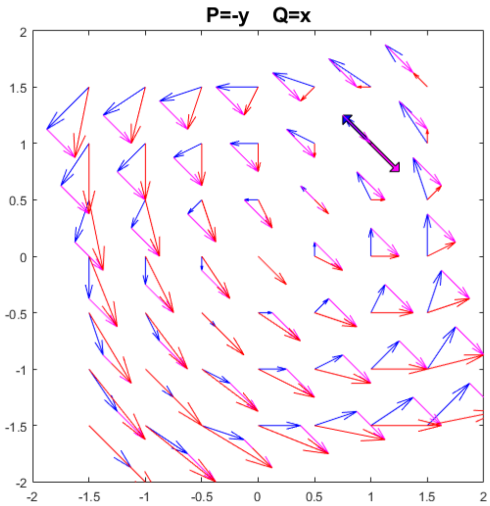
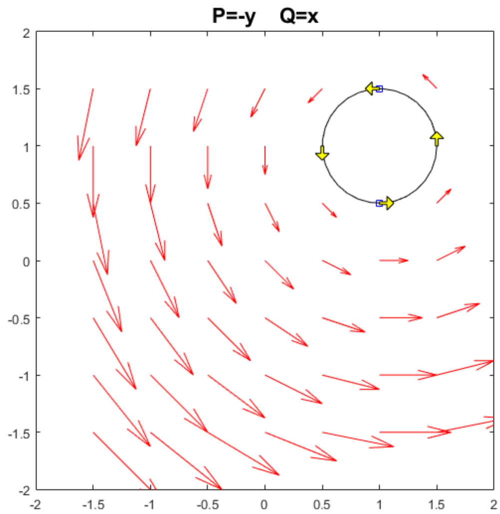
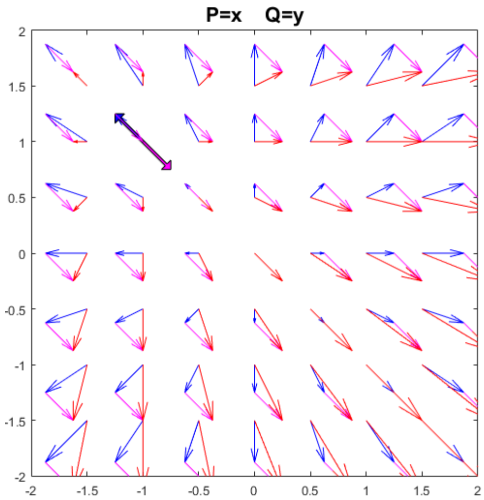
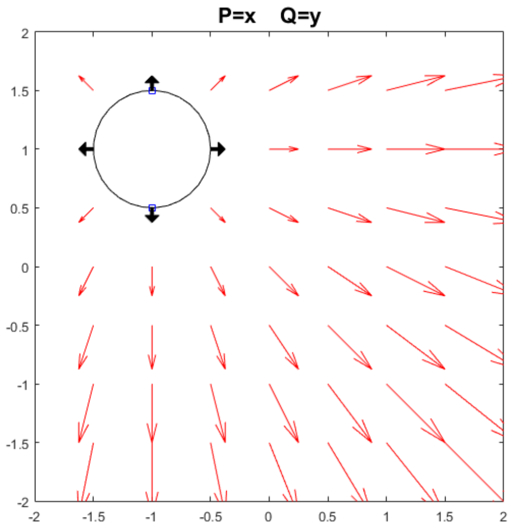

P=-y   Q=x


P=-y   Q=x


P=x   Q=y


P=x   Q=y