

Appl.Maths. B242-2023: LECTURE 7

LECTURE 7 [9.7] VECTOR FIELDS, DIV, CURL [1]

$$\underline{F}(x, y, z) = \begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix} = \underline{i} P(x, y, z) + \underline{j} Q(x, y, z) + \underline{k} R(x, y, z)$$

Vector field

2D

Example:

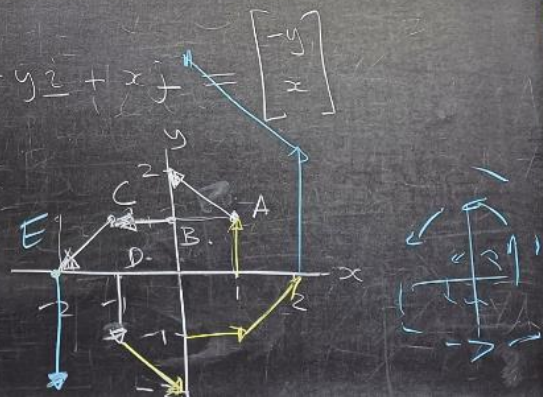
$$\underline{F} = -y\underline{i} + x\underline{j} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

E has 2 comp  
depend (x, y)

$$A: (1, 1) \quad \underline{F} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$B: (0, 1) \quad \underline{F} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$C: (-1, 1) \quad \underline{F} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

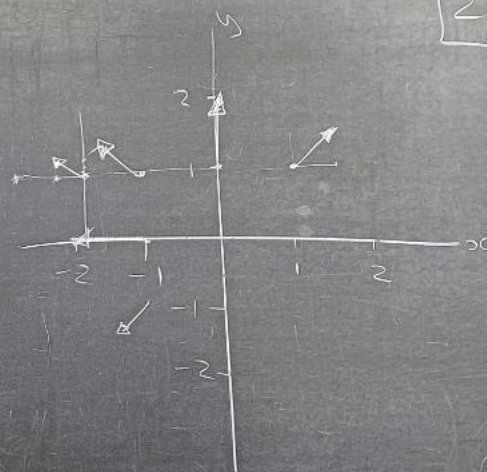


$$D: (-1, 0) \quad \underline{F} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$E: (-2, 0) \quad \underline{F} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Example:

$$\underline{F} = \begin{bmatrix} x \\ \frac{x^2+y^2}{y} \\ y \\ \frac{y}{x^2+y^2} \end{bmatrix}$$



# SUMMARY

3

$$r(t)$$



space curve

$$r'(t) \text{ --- tangent vector}$$

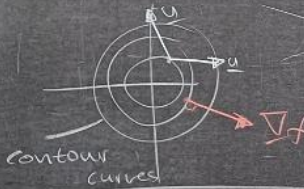
$$z = f(x, y)$$



surface

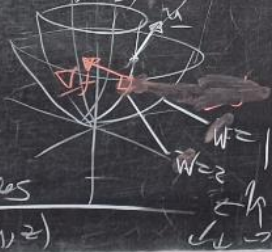
$$D_u f \Big|_{(x_0, y_0)}, \quad \nabla f \Big|_{(x_0, y_0)}$$

GRADIENT.



contour curves

$$W = f(x, y, z)$$



contour surfaces

$$F(x, y, z)$$

volume function.

$$D_u f \Big|_{(x_0, y_0, z_0)}, \quad \nabla f \Big|_{(x_0, y_0, z_0)}$$

# DIVERGENCE

4

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

$$\begin{aligned} \nabla \cdot \underline{F} &= \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \left(\frac{\partial}{\partial x}\right)P + \left(\frac{\partial}{\partial y}\right)Q + \left(\frac{\partial}{\partial z}\right)R \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \\ &= P_x + Q_y + R_z \end{aligned}$$

Subscript notation

$$\frac{\partial f}{\partial x} = f_x, \quad \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

Example: Calculate  $\nabla \cdot \underline{F}$  for  $\underline{F} = \begin{bmatrix} xyz \\ y^2ze^x \\ 2x+z^3 \end{bmatrix}$



$$\nabla \cdot \underline{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} xyz \\ y^2ze^x \\ 2x+z^3 \end{bmatrix} = yz + 2yze^x + 3z^2 \quad [5]$$

CURL

$$\nabla \times \underline{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{bmatrix}$$

Example: Find curl of  $\underline{F} = xyz \underline{i} + y^2ze^x \underline{j} + (2x+z^3) \underline{k}$

$$\nabla \times \underline{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} xyz \\ y^2ze^x \\ 2x+z^3 \end{bmatrix} = \begin{bmatrix} 0 - y^2e^x \\ xy - 2 \\ y^2ze^x - xz \end{bmatrix}$$

THREE DIFFERENTIAL OPERATORS

[6]

Gradient

$$\nabla f \rightarrow \text{vector field}$$

↑  
scalar func

$\nabla f$  point in dir of maximum increase

$\nabla f$ , "del f", "gradient of f", "grad f"

Divergence

$$\nabla \cdot \underline{F} \rightarrow \text{scalar function}$$

↑  
vector field

$\nabla \cdot \underline{F}$  "del dot F"  
"divergence of F", "div F"

Curl

$$\nabla \times \underline{F} \rightarrow \text{vector field}$$

↑  
vector field

$\nabla \times \underline{F}$  "del cross F" "Curl F"

# Flux of $\underline{F}$ through a surface

7

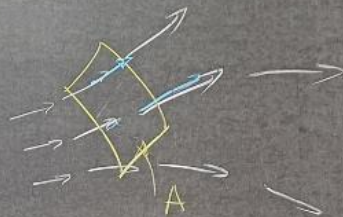
Fluid velocity

$\underline{F}$  in  $\text{m}\cdot\text{s}^{-1}$

Flux of  $\underline{F}$  in  $(\text{m}\cdot\text{s}^{-1})(\text{m}^2)$

$\text{m}^3\cdot\text{s}^{-1}$

volume per time



$\text{Flux} = \|\underline{F}_\perp\| \cdot A$

$\text{Flux} = \underline{F} \cdot \underline{n} \cdot A$

$= \|\underline{F}\| \cdot A \cdot (\cos \theta)$

$= F_\perp \cdot A$



$\text{flux} \int_{\text{surface}} \underline{F} \cdot \underline{n} \, dA$

# Electric field

8

$\underline{E}$  in  $\text{V}/\text{m}$

flux of  $\underline{E}$  in  $\text{Vm}$



