

Appl. Maths. B242-2023: LECTURE 6

**LECTURE 6** GRADIENT, [9.6] TANGENT PLANES ①

Dir. der. 2D

$$D_u f(x,y) \Big|_{(x_0, y_0)}$$

Make a path

$$x = x_0 + t \cos \theta$$

$$y = y_0 + t \sin \theta$$

Sample  $f(x,y)$  along the path, call it  $g(t)$

$$g(t) = f(x_0 + t \cos \theta, y_0 + t \sin \theta)$$

Slope along path (at  $(x_0, y_0)$ )

$g'(t)$  = *derivative* (A)

$$\lim_{\Delta t \rightarrow 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{f(x_0 + (t + \Delta t) \cos \theta, y_0 + (t + \Delta t) \sin \theta) - f(x_0 + t \cos \theta, y_0 + t \sin \theta)}{\Delta t}$$

chain rule

$$= \lim_{\Delta t \rightarrow 0} \frac{f(x_0 + \Delta t \cos \theta, y_0 + \Delta t \sin \theta) - f(x_0, y_0)}{\Delta t}$$

$$= D_u f(x,y) \Big|_{(x_0, y_0)}$$

(B)

$$\frac{dg}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

$$D_u f(x,y) \Big|_{(x_0, y_0)} = \frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} \cos \theta + \frac{\partial f}{\partial y} \Big|_{(x_0, y_0)} \sin \theta$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} = \nabla f \cdot \underline{u}$$

normalised

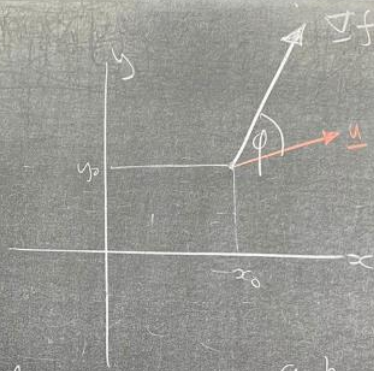
### GRADIENT IN 2D

$$z = f(x, y)$$

$\nabla f$  is a vector

$$D_u f = \nabla f \cdot \underline{u}$$

$$= \|\nabla f\| \|\underline{u}\| \cos \phi$$



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$$\underline{a} \cdot \underline{b} = ab \cos \theta = \|\underline{a}\| \|\underline{b}\| \cos \theta$$

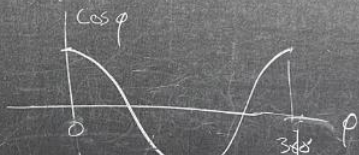
In which direction is  $D_u f$  maximal?

→ at  $\phi = 0$

$\nabla f$  is a vector

points in direction of maximum slope increase

$\|\nabla f\|$  is the rate at which  $f(x, y)$  increases.



### GRADIENT IN 3D

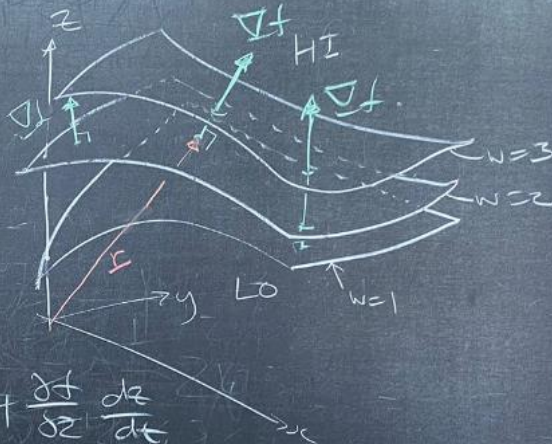
$$w = f(x, y, z)$$

$$\nabla f = \begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \\ \partial f / \partial z \end{bmatrix}$$

$$c = f(x, y, z)$$

$$\frac{dc}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

$$\odot = \begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \\ \partial f / \partial z \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \nabla f \cdot \underline{u}$$



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$$0 = \frac{x^2 + y^2}{3} - z$$

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$$W = \frac{x^2 + y^2}{3} - z = f(x, y, z)$$

$$\nabla W = \begin{bmatrix} 2x/3 \\ 2y/3 \\ -1 \end{bmatrix} \quad \nabla W \Big|_{\substack{x=1 \\ y=0.6}} = \begin{bmatrix} 0.6667 \\ 0.4 \\ -1 \end{bmatrix}$$

[9.6] TANGENT PLANES

Ex: Find tangent plane to  $\frac{x^2 + y^2}{z} = \text{const.}$

at the point  $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$\text{const} = \frac{1^2 + 1^2}{2} = 1$$

Surface:  $1 = \frac{x^2 + y^2}{z}$  rewrite  $z = x^2 + y^2$

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Plane:  $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$   
normal      a point

$$\vec{r}_0 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$W = \frac{x^2 + y^2}{z}$$

$$\nabla W = \begin{bmatrix} 2x/z \\ 2y/z \\ -\frac{x^2 + y^2}{z^2} \end{bmatrix} \quad \nabla W \Big|_{\substack{x=1 \\ y=1 \\ z=2}} = \begin{bmatrix} 2 \cdot 1/2 \\ 2 \cdot 1/2 \\ -\frac{(1+1)}{2^2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ -1/2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$x + y - \frac{1}{2}z = 1$$

