Exercise 9.9

9.9 Exercises Answers to selected odd-numbered problems begin on page ANS-21.

In Problems 1–10, show that the given line integral is independent of the path. Evaluate in two ways: **(a)** Find a potential function ϕ and then use Theorem 9.9.1, and **(b)** Use any convenient path between the endpoints of the path.

1. $\int_{(0,0)}^{(2,2)} x^2\,dx + y^2\,dy$

2. $\int_{(1,1)}^{(2,4)} 2xy\,dx + x^2\,dy$

3. $\int_{(1,0)}^{(3,2)} (x + 2y)\,dx + (2x - y)\,dy$

4. $\int_{(0,0)}^{(\pi/2,0)} \cos x \cos y\,dx + (1 - \sin x \sin y)\,dy$

5. $\int_{(4,1)}^{(4,4)} \dfrac{-y\,dx + x\,dy}{y^2}$ on any path not crossing the x-axis

6. $\int_{(1,0)}^{(3,4)} \dfrac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$ on any path not through the origin

7. $\int_{(1,2)}^{(3,6)} (2y^2x - 3)\,dx + (2yx^2 + 4)\,dy$

8. $\int_{(-1,1)}^{(0,0)} (5x + 4y)\,dx + (4x - 8y^3)\,dy$

9. $\int_{(0,0)}^{(2,8)} (y^3 + 3x^2y)\,dx + (x^3 + 3y^2x + 1)\,dy$

10. $\int_{(-2,0)}^{(1,0)} (2x - y\sin xy - 5y^4)\,dx - (20xy^3 + x\sin xy)\,dy$

In Problems 11–16, determine whether the given vector field is a conservative field. If so, find a potential function ϕ for **F**.

11. $\mathbf{F}(x, y) = (4x^3y^3 + 3)\mathbf{i} + (3x^4y^2 + 1)\mathbf{j}$

12. $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + 3y^2(x^2 + 1)\mathbf{j}$

13. $\mathbf{F}(x, y) = y^2 \cos xy^2\,\mathbf{i} - 2xy \sin xy^2\,\mathbf{j}$

14. $\mathbf{F}(x, y) = (x^2 + y^2 + 1)^{-2}(x\mathbf{i} + y\mathbf{j})$

15. $\mathbf{F}(x, y) = (x^3 + y)\mathbf{i} + (x + y^3)\mathbf{j}$

16. $\mathbf{F}(x, y) = 2e^{2y}\mathbf{i} + xe^{2y}\mathbf{j}$

In Problems 17 and 18, find the work done by the force $\mathbf{F}(x, y) = (2x + e^{-y})\mathbf{i} + (4y - xe^{-y})\mathbf{j}$ along the indicated curve.

17.

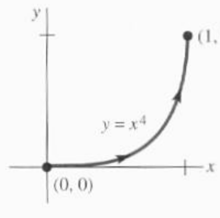

FIGURE 9.9.7 Curve for Problem 17

18.

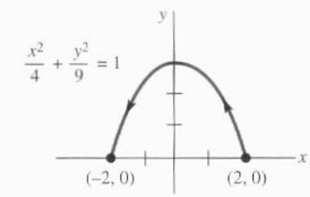

FIGURE 9.9.8 Curve for Problem 18

In Problems 19–24, show that the given integral is independent of the path. Evaluate.

19. $\int_{(1,1,1)}^{(2,4,8)} yz\,dx + xz\,dy + xy\,dz$

20. $\int_{(0,0,0)}^{(1,1,1)} 2x\,dx + 3y^2\,dy + 4z^3\,dz$

21. $\int_{(1,0,0)}^{(2,\pi/2,1)} (2x \sin y + e^{3z})\,dx + x^2 \cos y\,dy + (3xe^{3z} + 5)\,dz$

22. $\int_{(1,2,1)}^{(3,4,1)} (2x + 1)\,dx + 3y^2\,dy + \dfrac{1}{z}\,dz$

23. $\int_{(1,1,\ln 3)}^{(2,2,\ln 3)} e^{2z}\,dx + 3y^2\,dy + 2xe^{2z}\,dz$

24. $\int_{(-2,3,1)}^{(0,0,0)} 2xz\,dx + 2yz\,dy + (x^2 + y^2)\,dz$

In Problems 25 and 26, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

25. $\mathbf{F}(x, y, z) = (y - yz \sin x)\mathbf{i} + (x + z\cos x)\mathbf{j} + y\cos x\,\mathbf{k}$; $\mathbf{r}(t) = 2t\mathbf{i} + (1 + \cos t)^2\mathbf{j} + 4\sin^3 t\,\mathbf{k}$, $0 \le t \le \pi/2$

26. $\mathbf{F}(x, y, z) = (2 - e^z)\mathbf{i} + (2y - 1)\mathbf{j} + (2 - xe^z)\mathbf{k}$; $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $(-1, 1, -1)$ to $(2, 4, 8)$

27. The inverse square law of gravitational attraction between two masses m_1 and m_2 is given by $\mathbf{F} = -Gm_1m_2\mathbf{r}/\|\mathbf{r}\|^3$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that **F** is conservative. Find a potential function for **F**.

28. Find the work done by the force $\mathbf{F}(x, y, z) = 8xy^3z\,\mathbf{i} + 12x^2y^2z\,\mathbf{j} + 4x^2y^3\,\mathbf{k}$ acting along the helix $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\,\mathbf{k}$ from $(2, 0, 0)$ to $(1, \sqrt{3}, \pi/3)$. From $(2, 0, 0)$ to $(0, 2, \pi/2)$. [*Hint*: Show that **F** is conservative.]

29. If **F** is a conservative force field, show that the work done along any simple closed path is zero.

30. A particle in the plane is attracted to the origin with a force $\mathbf{F} = \|\mathbf{r}\|^n\mathbf{r}$, where n is a positive integer and $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is the position vector of the particle. Show that **F** is conservative. Find the work done in moving the particle between (x_1, y_1) and (x_2, y_2).

31. Suppose **F** is a conservative force field with potential function ϕ. In physics the function $p = -\phi$ is called **potential energy**. Since $\mathbf{F} = -\nabla p$, Newton's second law becomes

$$m\mathbf{r}'' = -\nabla p \quad \text{or} \quad m\frac{d\mathbf{v}}{dt} + \nabla p = \mathbf{0}.$$

By integrating $m\dfrac{d\mathbf{v}}{dt} \cdot \dfrac{d\mathbf{r}}{dt} + \nabla p \cdot \dfrac{d\mathbf{r}}{dt} = 0$ with respect to t, derive the law of conservation of mechanical energy: $\frac{1}{2}mv^2 + p = \text{constant}$. [*Hint*: See Problem 39 in Exercises 9.8.]

32. Suppose that C is a smooth curve between points A (at $t = a$) and B (at $t = b$) and that p is potential energy, defined in Problem 31. If **F** is a conservative force field and $K = \frac{1}{2}mv^2$ is kinetic energy, show that $p(B) + K(B) = p(A) + K(A)$.