# Exercise 9.7

## 9.7 Exercises

Answers to selected odd-numbered problems begin on page ANS-21.

In Problems 1–6, graph some representative vectors in the given vector field.

1. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$
2. $\mathbf{F}(x, y) = -x\mathbf{i} + y\mathbf{j}$
3. $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$
4. $\mathbf{F}(x, y) = x\mathbf{i} + 2y\mathbf{j}$
5. $\mathbf{F}(x, y) = y\mathbf{j}$
6. $\mathbf{F}(x, y) = x\mathbf{j}$

In Problems 7–16, find the curl and the divergence of the given vector field.

7. $\mathbf{F}(x, y, z) = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$
8. $\mathbf{F}(x, y, z) = 10yz\mathbf{i} + 2x^2z\mathbf{j} + 6x^3\mathbf{k}$
9. $\mathbf{F}(x, y, z) = 4xy\mathbf{i} + (2x^2 + 2yz)\mathbf{j} + (3z^2 + y^2)\mathbf{k}$
10. $\mathbf{F}(x, y, z) = (x - y)^3\mathbf{i} + e^{-yz}\mathbf{j} + xye^{2y}\mathbf{k}$
11. $\mathbf{F}(x, y, z) = 3x^2y\mathbf{i} + 2xz^3\mathbf{j} + y^4\mathbf{k}$
12. $\mathbf{F}(x, y, z) = 5y^3\mathbf{i} + (\frac{1}{2}x^3y^2 - xy)\mathbf{j} - (x^3yz - xz)\mathbf{k}$
13. $\mathbf{F}(x, y, z) = xe^{-z}\mathbf{i} + 4yz^2\mathbf{j} + 3ye^{-z}\mathbf{k}$
14. $\mathbf{F}(x, y, z) = yz \ln x\mathbf{i} + (2x - 3yz)\mathbf{j} + xy^2z^3\mathbf{k}$
15. $\mathbf{F}(x, y, z) = xye^x\mathbf{i} - x^3yze^z\mathbf{j} + xy^2e^y\mathbf{k}$
16. $\mathbf{F}(x, y, z) = x^2 \sin yz\mathbf{i} + z \cos xz^3\mathbf{j} + ye^{5xy}\mathbf{k}$

In Problems 17–24, let $\mathbf{a}$ be a constant vector and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Verify the given identity.

17. $\text{div}\,\mathbf{r} = 3$
18. $\text{curl}\,\mathbf{r} = \mathbf{0}$
19. $(\mathbf{a} \times \nabla) \times \mathbf{r} = -2\mathbf{a}$
20. $\nabla \times (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$
21. $\nabla \cdot (\mathbf{a} \times \mathbf{r}) = 0$
22. $\mathbf{a} \times (\nabla \times \mathbf{r}) = \mathbf{0}$
23. $\nabla \times [(\mathbf{r} \cdot \mathbf{r})\mathbf{a}] = 2(\mathbf{r} \times \mathbf{a})$
24. $\nabla \cdot [(\mathbf{r} \cdot \mathbf{r})\mathbf{a}] = 2(\mathbf{r} \cdot \mathbf{a})$

In Problems 25–32, verify the given identity. Assume continuity of all partial derivatives.

**25.** $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

**26.** $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

**27.** $\nabla \cdot (f\mathbf{F}) = f(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla f$

**28.** $\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F}$

**29.** $\text{curl}(\text{grad}\, f) = \mathbf{0}$

**30.** $\text{div}(\text{curl}\, \mathbf{F}) = 0$

**31.** $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl}\, \mathbf{F} - \mathbf{F} \cdot \text{curl}\, \mathbf{G}$

**32.** $\text{curl}(\text{curl}\, \mathbf{F} + \text{grad}\, f) = \text{curl}(\text{curl}\, \mathbf{F})$

**33.** Show that

$$\nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

This is known as the **Laplacian** and is also written $\nabla^2 f$.

**34.** Show that $\nabla \cdot (f\nabla f) = f\nabla^2 f + \|\nabla f\|^2$, where $\nabla^2 f$ is the Laplacian defined in Problem 33. [*Hint*: See Problem 27.]

**35.** Find curl(curl **F**) for the vector field $\mathbf{F} = xy\mathbf{i} + 4yz^2\mathbf{j} + 2xz\mathbf{k}$.

**36.** **(a)** Assuming continuity of all partial derivatives, show that $\text{curl}(\text{curl}\, \mathbf{F}) = -\nabla^2\mathbf{F} + \text{grad}(\text{div}\, \mathbf{F})$, where

$$\nabla^2\mathbf{F} = \nabla^2(P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) = \nabla^2 P\mathbf{i} + \nabla^2 Q\mathbf{j} + \nabla^2 R\mathbf{k}.$$

**(b)** Use the identity in part (a) to obtain the result in Problem 35.

**37.** Any scalar function $f$ for which $\nabla^2 f = 0$ is said to be **harmonic**. Verify that $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ is harmonic except at the origin. $\nabla^2 f = 0$ is called **Laplace's equation**.

**38.** Verify that

$$f(x, y) = \arctan\left(\frac{2}{x^2y^2 - 1}\right), \quad x^2 + y^2 \neq 1$$

satisfies Laplace's equation in two variables

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

**39.** Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a mass $m_1$ and let the mass $m_2$ be located at the origin. If the force of gravitational attraction is

$$\mathbf{F} = -\frac{Gm_1m_2}{\|\mathbf{r}\|^3}\mathbf{r},$$

verify that curl $\mathbf{F} = \mathbf{0}$ and div $\mathbf{F} = 0$, $\mathbf{r} \neq \mathbf{0}$.

**40.** Suppose a body rotates with a constant angular velocity $\boldsymbol{\omega}$ about an axis. If **r** is the position vector of a point $P$ on the body measured from the origin, then the linear velocity vector **v** of rotation is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. See **FIGURE 9.7.8**. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$, show that $\boldsymbol{\omega} = \frac{1}{2}\,\text{curl}\, \mathbf{v}$.

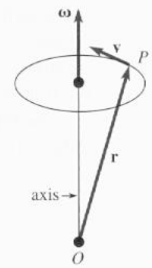


**FIGURE 9.7.8** Rotating body in Problem 40

In Problems 41 and 42, assume that $f$ and $g$ have continuous second partial derivatives. Show that the given vector field is solenoidal. [*Hint*: See Problem 31.]

**41.** $\mathbf{F} = \nabla f \times \nabla g$

**42.** $\mathbf{F} = \nabla f \times (f\nabla g)$

**43.** The velocity vector field for the two-dimensional flow of an ideal fluid around a cylinder is given by

$$\mathbf{F}(x, y) = A\left[\left(1 - \frac{x^2 - y^2}{(x^2 + y^2)^2}\right)\mathbf{i} - \frac{2xy}{(x^2 + y^2)^2}\mathbf{j}\right]$$

for some positive constant $A$. See **FIGURE 9.7.9**.

**(a)** Show that when the point $(x, y)$ is far from the origin, $\mathbf{F}(x, y) \approx A\mathbf{i}$.

**(b)** Show that **F** is irrotational.

**(c)** Show that **F** is incompressible.

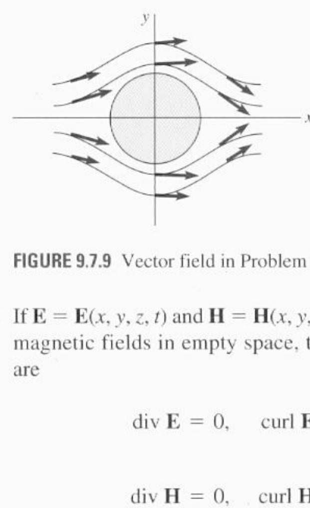


**FIGURE 9.7.9** Vector field in Problem 43

**44.** If $\mathbf{E} = \mathbf{E}(x, y, z, t)$ and $\mathbf{H} = \mathbf{H}(x, y, z, t)$ represent electric and magnetic fields in empty space, then Maxwell's equations are

$$\text{div}\, \mathbf{E} = 0, \quad \text{curl}\, \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t},$$

$$\text{div}\, \mathbf{H} = 0, \quad \text{curl}\, \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

where $c$ is the speed of light. Use the identity in Problem 36(a) to show that **E** and **H** satisfy

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2\mathbf{H} = \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}.$$

**45.** Consider the vector field $\mathbf{F} = x^2yz\mathbf{i} - xy^2z\mathbf{j} + (z + 5x)\mathbf{k}$. Explain why **F** is not the curl of another vector field **G**.