where ϕ is the angle between ∇f and $\mathbf{u}$. Because $0 \le \phi \le \pi$, we have $-1 \le \cos\phi \le 1$ and, consequently, $-\|\nabla f\| \le D_{\mathbf{u}} f \le \|\nabla f\|$. In other words:

The maximum value of the directional derivative is $\|\nabla f\|$ and it occurs when $\mathbf{u}$ has the same direction as ∇f (when $\cos\phi = 1$), (10)

and

The minimum value of the directional derivative is $-\|\nabla f\|$ and it occurs when $\mathbf{u}$ and ∇f have opposite directions (when $\cos\phi = -1$). (11)

EXAMPLE 6 Max/Min of Directional Derivative

In Example 5 the maximum value of the directional derivative of F of $(1, -1, 2)$ is $\|\nabla F(1, -1, 2)\| = \sqrt{133}$. The minimum value of $D_{\mathbf{u}}F(1, -1, 2)$ is then $-\sqrt{133}$.

Gradient Points in Direction of Most Rapid Increase of *f* Put yet another way, (10) and (11) state:

The gradient vector ∇f points in the direction in which f increases most rapidly, whereas $-\nabla f$ points in the direction of the most rapid decrease of f.

EXAMPLE 7 Direction of Steepest Ascent

Each year in Los Angeles there is a bicycle race up to the top of a hill by a road known to be the steepest in the city. To understand why a bicyclist with a modicum of sanity will zigzag up the road, let us suppose the graph of $f(x, y) = 4 - \frac{2}{3}\sqrt{x^2 + y^2}$, $0 \le z \le 4$, shown in **FIGURE 9.5.3(a)** is a mathematical model of the hill. The gradient of f is

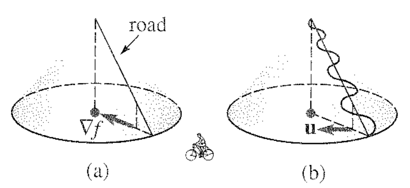

FIGURE 9.5.3 Model of a hill in Example 7

$$\nabla f(x, y) = \frac{2}{3}\left[\frac{-x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{-y}{\sqrt{x^2 + y^2}}\mathbf{j}\right] = \frac{\frac{2}{3}}{\sqrt{x^2 + y^2}}\mathbf{r},$$

where $\mathbf{r} = -x\mathbf{i} - y\mathbf{j}$ is a vector pointing to the center of the circular base.

Thus the steepest ascent up the hill is a straight road whose projection in the xy-plane is a radius of the circular base. Since $D_{\mathbf{u}} f = \text{comp}_{\mathbf{u}}\nabla f$, a bicyclist will zigzag, or seek a direction $\mathbf{u}$ other than ∇f, in order to reduce this component.

EXAMPLE 8 Direction to Cool Off Fastest

The temperature in a rectangular box is approximated by

$$T(x, y, z) = xyz(1 - x)(2 - y)(3 - z), \quad 0 \le x \le 1, \quad 0 \le y \le 2, \quad 0 \le z \le 3.$$

If a mosquito is located at $(\frac{1}{2}, 1, 1)$, in which direction should it fly to cool off as rapidly as possible?

Solution The gradient of T is

$$\nabla T(x, y, z) = yz(2 - y)(3 - z)(1 - 2x)\,\mathbf{i} + xz(1 - x)(3 - z)(2 - 2y)\,\mathbf{j} + xy(1 - x)(2 - y)(3 - 2z)\,\mathbf{k}.$$

Therefore, $\nabla T(\frac{1}{2}, 1, 1) = \frac{1}{4}\mathbf{k}$. To cool off most rapidly, the mosquito should fly in the direction of $-\frac{1}{4}\mathbf{k}$; that is, it should dive for the floor of the box, where the temperature is $T(x, y, 0) = 0$.

9.5 Exercises

Answers to selected odd-numbered problems begin on page ANS-20.

In Problems 1–4, compute the gradient for the given function.

1. $f(x, y) = x^2 - x^3y^2 + y^4$
2. $f(x, y) = y - e^{-2x^2y}$
3. $F(x, y, z) = \dfrac{xy^2}{z^3}$
4. $F(x, y, z) = xy\cos yz$

In Problems 5–8, find the gradient of the given function at the indicated point.

5. $f(x, y) = x^2 - 4y^2$; $(2, 4)$
6. $f(x, y) = \sqrt{x^3y - y^4}$; $(3, 2)$

7. $F(x, y, z) = x^2z^2 \sin 4y$; $(-2, \pi/3, 1)$
8. $F(x, y, z) = \ln(x^2 + y^2 + z^2)$; $(-4, 3, 5)$

In Problems 9 and 10, use Definition 9.5.1 to find $D_{\mathbf{u}}f(x, y)$ given that $\mathbf{u}$ makes the indicated angle with the positive x-axis.

9. $f(x, y) = x^2 + y^2$; $\theta = 30°$
10. $f(x, y) = 3x - y^2$; $\theta = 45°$

In Problems 11–20, find the directional derivative of the given function at the given point in the indicated direction.

11. $f(x, y) = 5x^3y^6$; $(-1, 1)$, $\theta = \pi/6$
12. $f(x, y) = 4x + xy^2 - 5y$; $(3, -1)$, $\theta = \pi/4$
13. $f(x, y) = \tan^{-1} \dfrac{y}{x}$; $(2, -2)$, $\mathbf{i} - 3\mathbf{j}$
14. $f(x, y) = \dfrac{xy}{x + y}$; $(2, -1)$, $6\mathbf{i} + 8\mathbf{j}$
15. $f(x, y) = (xy + 1)^2$; $(3, 2)$, in the direction of $(5, 3)$
16. $f(x, y) = x^2 \tan y$; $(\frac{1}{2}, \pi/3)$, in the direction of the negative x-axis
17. $F(x, y, z) = x^2y^2(2z + 1)^2$; $(1, -1, 1)$, $\langle 0, 3, 3\rangle$
18. $F(x, y, z) = \dfrac{x^2 - y^2}{z^2}$; $(2, 4, -1)$, $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$
19. $F(x, y, z) = \sqrt{x^2y + 2y^2z}$; $(-2, 2, 1)$, in the direction of the negative z-axis
20. $F(x, y, z) = 2x - y^2 + z^2$; $(4, -4, 2)$, in the direction of the origin

In Problems 21 and 22, consider the plane through the points P and Q that is perpendicular to the xy-plane. Find the slope of the tangent at the indicated point to the curve of intersection of this plane and the graph of the given function in the direction of Q.

21. $f(x, y) = (x - y)^2$; $P(4, 2)$, $Q(0, 1)$; $(4, 2, 4)$
22. $f(x, y) = x^3 - 5xy + y^2$; $P(1, 1)$, $Q(-1, 6)$; $(1, 1, -3)$

In Problems 23–26, find a vector that gives the direction in which the given function increases most rapidly at the indicated point. Find the maximum rate.

23. $f(x, y) = e^{2x} \sin y$; $(0, \pi/4)$
24. $f(x, y) = xye^{x-y}$; $(5, 5)$
25. $F(x, y, z) = x^2 + 4xz + 2yz^2$; $(1, 2, -1)$
26. $F(x, y, z) = xyz$; $(3, 1, -5)$

In Problems 27–30, find a vector that gives the direction in which the given function decreases most rapidly at the indicated point. Find the minimum rate.

27. $f(x, y) = \tan(x^2 + y^2)$; $(\sqrt{\pi/6}, \sqrt{\pi/6})$
28. $f(x, y) = x^3 - y^3$; $(2, -2)$
29. $F(x, y, z) = \sqrt{xze^y}$; $(16, 0, 9)$
30. $F(x, y, z) = \ln \dfrac{xy}{z}$; $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$
31. Find the directional derivative(s) of $f(x, y) = x + y^2$ at $(3, 4)$ in the direction of a tangent vector to the graph of $2x^2 + y^2 = 9$ at $(2, 1)$.
32. If $f(x, y) = x^2 + xy + y^2 - x$, find all points where $D_{\mathbf{u}}f(x, y)$ in the direction of $\mathbf{u} = (1/\sqrt{2})(\mathbf{i} + \mathbf{j})$ is zero.
33. Suppose $\nabla f(a, b) = 4\mathbf{i} + 3\mathbf{j}$. Find a unit vector $\mathbf{u}$ so that
    (a) $D_{\mathbf{u}}f(a, b) = 0$,
    (b) $D_{\mathbf{u}}f(a, b)$ is a maximum, and
    (c) $D_{\mathbf{u}}f(a, b)$ is a minimum.
34. Suppose $D_{\mathbf{u}}f(a, b) = 6$. What is the value of $D_{-\mathbf{u}}f(a, b)$?
35. (a) If $f(x, y) = x^3 - 3x^2y^2 + y^3$, find the directional derivative of f at a point (x, y) in the direction of $\mathbf{u} = (1/\sqrt{10})(3\mathbf{i} + \mathbf{j})$.
    (b) If $F(x, y) = D_{\mathbf{u}}f(x, y)$ in part (a), find $D_{\mathbf{u}}F(x, y)$.
36. Consider the gravitational potential

$$U(x, y) = \frac{-Gm}{\sqrt{x^2 + y^2}},$$

where G and m are constants. Show that U increases or decreases most rapidly along a line through the origin.

37. If $f(x, y) = x^3 - 12x + y^2 - 10y$, find all points at which $\|\nabla f\| = 0$.
38. Suppose

$$D_{\mathbf{u}}f(a, b) = 7, \quad D_{\mathbf{v}}f(a, b) = 3$$

$$\mathbf{u} = \frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j}, \quad \mathbf{v} = \frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}.$$

Find $\nabla f(a, b)$.

39. Consider the rectangular plate shown in **FIGURE 9.5.4**. The temperature at a point (x, y) on the plate is given by $T(x, y) = 5 + 2x^2 + y^2$. Determine the direction an insect should take, starting at $(4, 2)$, in order to cool off as rapidly as possible.

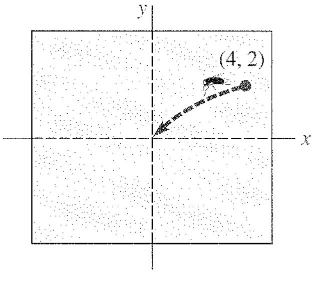

FIGURE 9.5.4 Insect in Problem 39

40. In Problem 39, observe that $(0, 0)$ is the coolest point of the plate. Find the path the cold-seeking insect, starting at $(4, 2)$, will take to the origin. If $\langle x(t), y(t)\rangle$ is the vector equation of the path, then use the fact that $-\nabla T(x, y) = \langle x'(t), y'(t)\rangle$. Why is this? [*Hint*: Remember separation of variables?]
41. The temperature at a point (x, y) on a rectangular metal plate is given by $T(x, y) = 100 - 2x^2 - y^2$. Find the path a heat-seeking particle will take, starting at $(3, 4)$, as it moves in the direction in which the temperature increases most rapidly.
42. The temperature T at a point (x, y, z) in space is inversely proportional to the square of the distance from (x, y, z) to the origin. It is known that $T(0, 0, 1) = 500$. Find the rate of change of T at $(2, 3, 3)$ in the direction of $(3, 1, 1)$. In which direction from $(2, 3, 3)$ does the temperature T increase most rapidly? At $(2, 3, 3)$ what is the maximum rate of change of T?
43. Find a function f such that

$$\nabla f = (3x^2 + y^3 + ye^{xy})\mathbf{i} + (-2y^2 + 3xy^2 + xe^{xy})\mathbf{j}.$$

44. Let f_x, f_y, f_{xy}, f_{yx} be continuous and $\mathbf{u}$ and $\mathbf{v}$ be unit vectors. Show that $D_{\mathbf{u}}D_{\mathbf{v}}f = D_{\mathbf{v}}D_{\mathbf{u}}f$.

In Problems 45–48, assume that f and g are differentiable functions of two variables. Prove the given identity.

45. $\nabla(cf) = c\,\nabla f$ **46.** $\nabla(f + g) = \nabla f + \nabla g$

47. $\nabla(fg) = f\nabla g + g\nabla f$ **48.** $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$

49. If $\mathbf{F}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$ and

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

show that

$$\nabla \times \mathbf{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}.$$

9.6 Tangent Planes and Normal Lines

Introduction The notion of the gradient of a function of two or more variables was introduced in the preceding section as an aid in computing directional derivatives. In this section we give a geometric interpretation of the gradient vector.

Geometric Interpretation of the Gradient—Functions of Two Variables Suppose $f(x, y) = c$ is the level curve of the differential function $z = f(x, y)$ that passes through a specified point $P(x_0, y_0)$; that is, $f(x_0, y_0) = c$.

If this level curve is parameterized by the differentiable functions

$$x = g(t),\ y = h(t) \qquad \text{such that} \qquad x_0 = g(t_0),\ y_0 = h(t_0),$$

then the derivative of $f(g(t), h(t)) = c$ with respect to t is

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0. \tag{1}$$

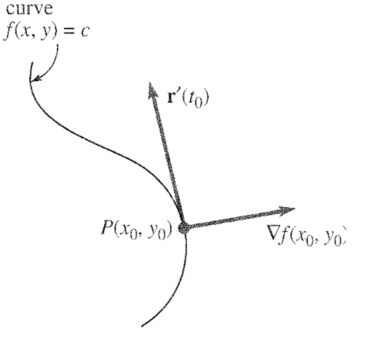

FIGURE 9.6.1 Gradient is perpendicular to tangent vector at P

When we introduce the vectors

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \qquad \text{and} \qquad \mathbf{r}'(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j},$$

(1) becomes $\nabla f \cdot \mathbf{r}' = 0$. Specifically, at $t = t_0$, we have

$$\nabla f(x_0, y_0) \cdot \mathbf{r}'(t_0) = 0. \tag{2}$$

Thus, if $\mathbf{r}'(t_0) \neq 0$, the vector $\nabla f(x_0, y_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ at $P(x_0, y_0)$. We interpret this to mean

∇f is orthogonal to the level curve at P.

See **FIGURE 9.6.1**.

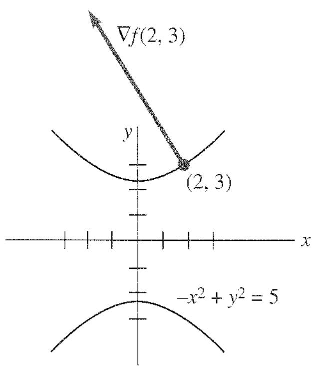

FIGURE 9.6.2 Gradient in Example 1

EXAMPLE 1 Gradient at a Point

Find the level curve of $f(x, y) = -x^2 + y^2$ passing through (2, 3). Graph the gradient at the point.

Solution Since $f(2, 3) = -4 + 9 = 5$, the level curve is the hyperbola $-x^2 + y^2 = 5$. Now,

$$\nabla f(x, y) = -2x\mathbf{i} + 2y\mathbf{j} \qquad \text{and} \qquad \nabla f(2, 3) = -4\mathbf{i} + 6\mathbf{j}.$$

FIGURE 9.6.2 shows the level curve and $\nabla f(2, 3)$. ≡

Geometric Interpretation of the Gradient—Functions of Three Variables Proceeding as before, let $F(x, y, z) = c$ be the *level surface* of a differentiable function $w = F(x, y, z)$ that passes through $P(x_0, y_0, z_0)$. If the differentiable functions $x = f(t)$, $y = g(t)$, $z = h(t)$ are the parametric equations of a curve C on the surface for which $x_0 = f(t_0)$, $y_0 = g(t_0)$, $z_0 = h(t_0)$, then the derivative of $F(f(t), g(t), h(t)) = 0$ implies that

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$