Remarks

If $w = F(x, y, z)$ has continuous partial derivatives of any order, then analogous to (3), the mixed partial derivatives are equal:

$$F_{xyz} = F_{yzx} = F_{zyx}, \qquad F_{xxy} = F_{yxx} = F_{xyx},$$

and so on.

9.4 Exercises

Answers to selected odd-numbered problems begin on page ANS-20.

In Problems 1–6, sketch some of the level curves associated with the given function.

1. $f(x, y) = x + 2y$

2. $f(x, y) = y^2 - x$

3. $f(x, y) = \sqrt{x^2 - y^2 - 1}$

4. $f(x, y) = \sqrt{36 - 4x^2 - 9y^2}$

5. $f(x, y) = e^{y - x^2}$

6. $f(x, y) = \tan^{-1}(y - x)$

In Problems 7–10, describe the level surfaces but do not graph.

7. $F(x, y, z) = \dfrac{x^2}{9} + \dfrac{z^2}{4}$

8. $F(x, y, z) = x^2 + y^2 + z^2$

9. $F(x, y, z) = x^2 + 3y^2 + 6z^2$

10. $F(x, y, z) = 4y - 2z + 1$

11. Graph some of the level surfaces associated with $F(x, y, z) = x^2 + y^2 - z^2$ for $c = 0$, $c > 0$, and $c < 0$.

12. Given that

$$F(x, y, z) = \frac{x^2}{16} + \frac{y^2}{4} + \frac{z^2}{9},$$

find the x-, y-, and z-intercepts of the level surface that passes through $(-4, 2, -3)$.

In Problems 13–32, find the first partial derivatives of the given function.

13. $z = x^2 - xy^2 + 4y^5$

14. $z = -x^3 + 6x^2y^3 + 5y^2$

15. $z = 5x^4y^3 - x^2y^6 + 6x^5 - 4y$

16. $z = \tan(x^3y^2)$

17. $z = \dfrac{4\sqrt{x}}{3y^2 + 1}$

18. $z = 4x^3 - 5x^2 + 8x$

19. $z = (x^3 - y^2)^{-1}$

20. $z = (-x^4 + 7y^2 + 3y)^6$

21. $z = \cos^2 5x + \sin^2 5y$

22. $z = e^{x^2 \tan^{-1} y^2}$

23. $f(x, y) = xe^{x^3 y}$

24. $f(\theta, \phi) = \phi^2 \sin \dfrac{\theta}{\phi}$

25. $f(x, y) = \dfrac{3x - y}{x + 2y}$

26. $f(x, y) = \dfrac{xy}{(x^2 - y^2)^2}$

27. $g(u, v) = \ln(4u^2 + 5v^3)$

28. $h(r, s) = \dfrac{\sqrt{r}}{s} - \dfrac{\sqrt{s}}{r}$

29. $w = 2\sqrt{xy} - ye^{y/z}$

30. $w = xy \ln xz$

31. $F(u, v, x, t) = u^2w^2 - uv^3 + vw\cos(ut^2) + (2x^2t)^4$

32. $G(p, q, r, s) = (p^2q^3)^{r^4s^5}$

In Problems 33 and 34, verify that the given function satisfies **Laplace's equation**:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

33. $z = \ln(x^2 + y^2)$

34. $z = e^{x^2 - y^2} \cos 2xy$

In Problems 35 and 36 verify that the given function satisfies the **wave equation**:

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

35. $u = \cos at \sin x$

36. $u = \cos(x + at) + \sin(x - at)$

37. The molecular concentration $C(x, t)$ of a liquid is given by $C(x, t) = t^{-1/2} e^{-x^2/kt}$. Verify that this function satisfies the **diffusion equation**:

$$\frac{k}{4} \frac{\partial^2 C}{\partial x^2} = \frac{\partial C}{\partial t}.$$

38. The pressure P exerted by an enclosed ideal gas is given by $P = k(T/V)$, where k is a constant, T is temperature, and V is volume. Find:

(a) the rate of change of P with respect to V,

(b) the rate of change of V with respect to T, and

(c) the rate of change of T with respect to P.

In Problems 39–48, use the Chain Rule to find the indicated partial derivatives.

39. $z = e^{uv^2}$; $u = x^3$, $v = x - y^2$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

40. $z = u^2 \cos 4v$; $u = x^2y^3$, $v = x^3 + y^3$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

41. $z = 4x - 5y^2$; $x = u^4 - 8v^3$, $y = (2u - v)^2$; $\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}$

42. $z = \dfrac{x - y}{x + y}$; $x = \dfrac{u}{v}$, $y = \dfrac{v^2}{u}$; $\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}$

43. $w = (u^2 + v^2)^{3/2}$; $u = e^{-t} \sin \theta$, $v = e^{-t} \cos \theta$; $\dfrac{\partial w}{\partial t}, \dfrac{\partial w}{\partial \theta}$

44. $w = \tan^{-1}\sqrt{uv}$; $u = r^2 - s^2$, $v = r^2s^2$; $\dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial s}$

45. $R = rs^2t^4$; $r = ue^{v^2}$, $s = ve^{-u^2}$, $t = e^{u^2v^2}$; $\dfrac{\partial R}{\partial u}, \dfrac{\partial R}{\partial v}$

46. $Q = \ln(pqr)$; $p = t^2 \sin^{-1} x$, $q = \dfrac{x}{t^2}$, $r = \tan^{-1} \dfrac{x}{t}$; $\dfrac{\partial Q}{\partial x}, \dfrac{\partial Q}{\partial t}$

47. $w = \sqrt{x^2 + y^2}$; $x = \ln(rs + tu)$,
$y = \frac{t}{u}\cosh rs$; $\frac{\partial w}{\partial t}, \frac{\partial w}{\partial r}, \frac{\partial w}{\partial u}$

48. $s = p^2 + q^2 - r^2 + 4t$; $p = \phi e^{3\theta}$, $q = \cos(\phi + \theta)$, $r = \phi\theta^2$,
$t = 2\phi + 8\theta$; $\frac{\partial s}{\partial \phi}, \frac{\partial s}{\partial \theta}$

In Problems 49–52, use (8) to find the indicated derivative.

49. $z = \ln(u^2 + v^2)$; $u = t^2$, $v = t^{-2}$; $\frac{dz}{dt}$

50. $z = u^3v - uv^4$; $u = e^{-5t}$, $v = \sec 5t$; $\frac{dz}{dt}$

51. $w = \cos(3u + 4v)$; $u = 2t + \frac{\pi}{2}$, $v = -t - \frac{\pi}{4}$; $\left.\frac{dw}{dt}\right|_{t=\pi}$

52. $w = e^{xy}$; $x = \frac{4}{2t + 1}$, $y = 3t + 5$; $\left.\frac{dw}{dt}\right|_{t=0}$

53. If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, show that **Laplace's equation** $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = 0$ becomes

$$\frac{\partial^2u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2} = 0.$$

54. Van der Waals' equation of state for the real gas CO_2 is

$$P = \frac{0.08T}{V - 0.0427} - \frac{3.6}{V^2}.$$

If dT/dt and dV/dt are rates at which the temperature and volume change, respectively, use the Chain Rule to find dP/dt.

55. The equation of state for a thermodynamic system is $F(P, V, T) = 0$, where P, V, and T are pressure, volume, and temperature, respectively. If the equation defines V as a function of P and T, and also defines T as a function of V and P, show that

$$\frac{\partial V}{\partial T} = -\frac{\frac{\partial F}{\partial T}}{\frac{\partial F}{\partial V}} = -\frac{1}{\frac{\partial T}{\partial V}}.$$

56. The voltage across a conductor is increasing at a rate of 2 volts/min and the resistance is decreasing at a rate of 1 ohm/min. Use $I = E/R$ and the Chain Rule to find the rate at which the current passing through the conductor is changing when $R = 50$ ohms and $E = 60$ volts.

57. The length of the side labeled x of the triangle in **FIGURE 9.4.6** increases at a rate of 0.3 cm/s, the side labeled y increases at a rate of 0.5 cm/s, and the included angle θ increases at a rate of 0.1 rad/s. Use the Chain Rule to find the rate at which the area of the triangle is changing at the instant $x = 10$ cm, $y = 8$ cm, and $\theta = \pi/6$.

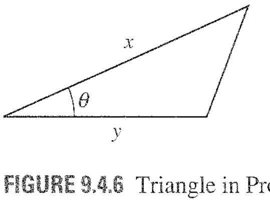

FIGURE 9.4.6 Triangle in Problem 57

58. A particle moves in 3-space so that its coordinates at any time are $x = 4\cos t$, $y = 4\sin t$, $z = 5t$, $t \geq 0$. Use the Chain Rule to find the rate at which its distance

$$w = \sqrt{x^2 + y^2 + z^2}$$

from the origin is changing at $t = 5\pi/2$ seconds.

9.5 Directional Derivative

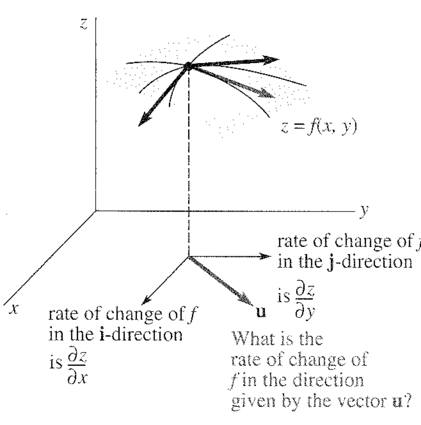

FIGURE 9.5.1 An arbitrary direction is denoted by the vector **u**

Introduction We saw in the last section that for a function f of two variables x and y, the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ give the slope of the tangent to the trace, or curve of intersection of the surface defined by $z = f(x, y)$ and vertical planes that are, respectively, parallel to the x- and y-coordinates axes. Equivalently, we can think of the partial derivative $\partial z/\partial x$ as the rate of change of the function f in the direction given by the vector **i**, and $\partial z/\partial y$ as the rate of change of the function f in the **j**-direction. There is no reason to confine our attention to just two directions. In this section we shall see how to find the rate of change of a differentiable function in *any* direction. See **FIGURE 9.5.1**.

The Gradient of a Function In this and the next sections it is convenient to introduce a new vector based on partial differentiation. When the **vector differential operator**

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} \quad \text{or} \quad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

is applied to a differentiable function $z = f(x, y)$ or $w = F(x, y, z)$, we say that the vectors

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \tag{1}$$

$$\nabla F(x, y, z) = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k} \tag{2}$$