# Exercise 9.16

## 9.16 Exercises

Answers to selected odd-numbered problems begin on page ANS-22.

In Problems 1 and 2, verify the divergence theorem.

1. $\mathbf{F} = xy\mathbf{i} + yz\,\mathbf{j} + xz\mathbf{k}$; $D$ the region bounded by the unit cube defined by $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$
2. $\mathbf{F} = 6xy\mathbf{i} + 4yz\mathbf{j} + xe^{-y}\mathbf{k}$; $D$ the region bounded by the three coordinate planes and the plane $x + y + z = 1$

In Problems 3–14, use the divergence theorem to find the outward flux $\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS$ of the given vector field $\mathbf{F}$.

3. $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$; $D$ the region bounded by the sphere $x^2 + y^2 + z^2 = a^2$
4. $\mathbf{F} = 4x\mathbf{i} + y\mathbf{j} + 4z\mathbf{k}$; $D$ the region bounded by the sphere $x^2 + y^2 + z^2 = 4$
5. $\mathbf{F} = y^2\mathbf{i} + xz^3\mathbf{j} + (z-1)^2\mathbf{k}$; $D$ the region bounded by the cylinder $x^2 + y^2 = 16$ and the planes $z = 1, z = 5$
6. $\mathbf{F} = x^2\mathbf{i} + 2yz\mathbf{j} + 4z^3\mathbf{k}$; $D$ the region bounded by the parallelepiped defined by $0 \le x \le 1, 0 \le y \le 2, 0 \le z \le 3$
7. $\mathbf{F} = y^3\mathbf{i} + x^3\mathbf{j} + z^3\mathbf{k}$; $D$ the region bounded within by $z = \sqrt{4 - x^2 - y^2}, x^2 + y^2 = 3, z = 0$
8. $\mathbf{F} = (x^2 + \sin y)\mathbf{i} + z^2\mathbf{j} + xy^3\mathbf{k}$; $D$ the region bounded by $y = x^2, z = 9 - y, z = 0$
9. $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\,\mathbf{k})/(x^2 + y^2 + z^2)$; $D$ the region bounded by the concentric spheres $x^2 + y^2 + z^2 = a^2, x^2 + y^2 + z^2 = b^2$, $b > a$
10. $\mathbf{F} = 2yz\mathbf{i} + x^3\mathbf{j} + xy^2\mathbf{k}$; $D$ the region bounded by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$
11. $\mathbf{F} = 2xz\mathbf{i} + 5y^2\mathbf{j} - z^2\mathbf{k}$; $D$ the region bounded by $z = y$, $z = 4 - y, z = 2 - \frac{1}{2}x^2, x = 0, z = 0$. See **FIGURE 9.16.6**.

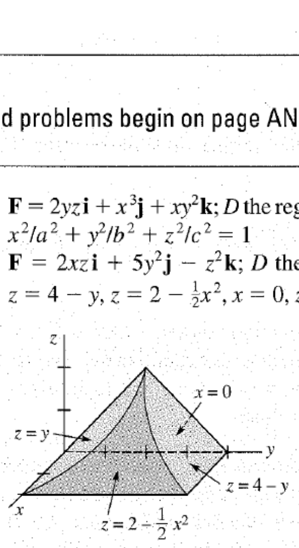


**FIGURE 9.16.6** Region $D$ for Problem 11

12. $\mathbf{F} = 15x^2y\mathbf{i} + x^2z\,\mathbf{j} + y^4\mathbf{k}$; $D$ the region bounded by $x + y = 2$, $z = x + y, z = 3, x = 0, y = 0$
13. $\mathbf{F} = 3x^2y^2\mathbf{i} + y\mathbf{j} - 6zxy^2\mathbf{k}$; $D$ the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$
14. $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + 6\sin x\mathbf{k}$; $D$ the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the planes $z = 2, z = 4$

15. The electric field at a point $P(x, y, z)$ due to a point charge $q$ located at the origin is given by the inverse square field

$$E = q\frac{\mathbf{r}}{\|\mathbf{r}\|^3},$$

where $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

**(a)** Suppose $S$ is a closed surface, $S_a$ is a sphere $x^2 + y^2 + z^2 = a^2$ lying completely within $S$, and $D$ is the region bounded between $S$ and $S_a$. See **FIGURE 9.16.7**. Show that the outward flux of $\mathbf{E}$ for the region $D$ is zero.

**(b)** Use the result of part (a) to prove **Gauss' law**:

$$\iint_S (\mathbf{E} \cdot \mathbf{n})\, dS = 4\pi q;$$

that is, the outward flux of the electric field $\mathbf{E}$ through *any* closed surface (for which the divergence theorem applies) containing the origin is $4\pi q$.

In Problems 17–21, assume that $S$ forms the boundary of a closed and bounded region $D$.

17. If $\mathbf{a}$ is a constant vector, show that $\iint_S (\mathbf{a} \cdot \mathbf{n})\, dS = 0$.
18. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and $P$, $Q$, and $R$ have continuous second partial derivatives, prove that

$$\iint_S (\text{curl } \mathbf{F} \cdot \mathbf{n})\, dS = 0.$$

In Problems 19 and 20, assume that $f$ and $g$ are scalar functions with continuous second partial derivatives. Use the divergence theorem to establish **Green's identities**.

19. $\displaystyle\iint_S (f\nabla g) \cdot \mathbf{n}\, dS = \iiint_D (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV$
20. $\displaystyle\iint_S (f\nabla g - g\nabla f) \cdot \mathbf{n}\, dS = \iiint_D (f\nabla^2 g - g\nabla^2 f)\, dV$
21. If $f$ is a scalar function with continuous first partial derivatives, prove that

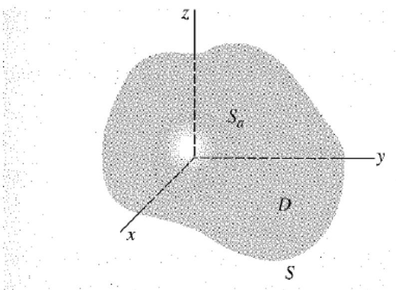


**FIGURE 9.16.7** Region $D$ for Problem 15(a)

**16.** Suppose there is a continuous distribution of charge throughout a closed and bounded region $D$ enclosed by a surface $S$. Then the natural extension of Gauss' law is given by

$$\iint_S (\mathbf{E} \cdot \mathbf{n})\, dS = \iiint_D 4\pi\rho\, dV,$$

where $\rho(x, y, z)$ is the charge density or charge per unit volume.

**(a)** Proceed as in the derivation of the continuity equation (16) to show that div $\mathbf{E} = 4\pi\rho$.

**(b)** Given that $\mathbf{E}$ is an irrotational vector field, show that the potential $\phi$ for $\mathbf{E}$ satisfies Poisson's equation $\nabla^2\phi = 4\pi\rho$.

prove that

$$\iint_S f\mathbf{n}\, dS = \iiint_D \nabla f\, dV.$$

[*Hint*: Use (2) on $f\mathbf{a}$, where $\mathbf{a}$ is a constant vector, and Problem 27 in Exercises 9.7.]

**22.** The buoyancy force on a floating object is $\mathbf{B} = -\iint_S p\,\mathbf{n}\, dS$, where $p$ is the fluid pressure. The pressure $p$ is related to the density of the fluid $\rho(x, y, z)$ by a law of hydrostatics: $\nabla p = \rho(x, y, z)\mathbf{g}$, where $\mathbf{g}$ is the constant acceleration of gravity. If the weight of the object is $\mathbf{W} = m\mathbf{g}$, use the result of Problem 21 to prove Archimedes' principle, $\mathbf{B} + \mathbf{W} = \mathbf{0}$. See **FIGURE 9.16.8**.

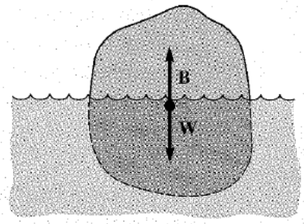


**FIGURE 9.16.8** Floating object in Problem 22