Exercise 9.13

9.13 Exercises

Answers to selected odd-numbered problems begin on page ANS-22.

1. Find the surface area of that portion of the plane $2x + 3y + 4z = 12$ that is bounded by the coordinate planes in the first octant.
2. Find the surface area of that portion of the plane $2x + 3y + 4z = 12$ that is above the region in the first quadrant bounded by the graph $r = \sin 2\theta$.
3. Find the surface area of that portion of the cylinder $x^2 + z^2 = 16$ that is above the region in the first quadrant bounded on the graphs of $x = 0$, $x = 2$, $y = 0$, $y = 5$.
4. Find the surface area of that portion of the paraboloid $z = x^2 + y^2$ that is below the plane $z = 2$.
5. Find the surface area of that portion of the paraboloid $z = 4 - x^2 - y^2$ that is above the xy-plane.
6. Find the surface area of those portions of the sphere $x^2 + y^2 + z^2 = 2$ that are within the cone $z^2 = x^2 + y^2$.
7. Find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = 25$ that is above the region in the first quadrant bounded by the graphs of $x = 0$, $y = 0$, $4x^2 + y^2 = 25$. [*Hint*: Integrate first with respect to x.]
8. Find the surface area of that portion of the graph of $z = x^2 - y^2$ that is in the first octant within the cylinder $x^2 + y^2 = 4$.
9. Find the surface area of the portions of the sphere $x^2 + y^2 + z^2 = a^2$ that are within the cylinder $x^2 + y^2 = ay$.
10. Find the surface area of the portions of the cone $z^2 = \frac{1}{4}(x^2 + y^2)$ that are within the cylinder $(x - 1)^2 + y^2 = 1$.
11. Find the surface area of the portions of the cylinder $y^2 + z^2 = a^2$ that are within the cylinder $x^2 + y^2 = a^2$. [*Hint*: See Figure 9.10.12.]
12. Use the result given in Example 1 to prove that the surface area of a sphere of radius a is $4\pi a^2$. [*Hint*: Consider a limit as $b \to a$.]
13. Find the surface area of that portion of the sphere $x^2 + y^2 + z^2 = a^2$ that is bounded between $y = c_1$ and $y = c_2$, $0 < c_1 < c_2 < a$. [*Hint*: Use polar coordinates in the xz-plane.]
14. Show that the area found in Problem 13 is the same as the surface area of the cylinder $x^2 + z^2 = a^2$ between $y = c_1$ and $y = c_2$.

In Problems 15–24, evaluate the surface integral $\iint_S G(x, y, z)\, dS$.

15. $G(x, y, z) = x$; S the portion of the cylinder $z = 2 - x^2$ in the first octant bounded by $x = 0$, $y = 0$, $y = 4$, $z = 0$
16. $G(x, y, z) = xy(9 - 4z)$; same surface as in Problem 15
17. $G(x, y, z) = xz^3$; S the cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 1$
18. $G(x, y, z) = x + y + z$; S the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 4$
19. $G(x, y, z) = (x^2 + y^2)z$; S that portion of the sphere $x^2 + y^2 + z^2 = 36$ in the first octant
20. $G(x, y, z) = z^2$; S that portion of the plane $z = x + 1$ within the cylinder $y = 1 - x^2$, $0 \le y \le 1$
21. $G(x, y, z) = xy$; S that portion of the paraboloid $2z = 4 - x^2 - y^2$ within $0 \le x \le 1$, $0 \le y \le 1$
22. $G(x, y, z) = 2z$; S that portion of the paraboloid $2z = 1 + x^2 + y^2$ in the first octant bounded by $x = 0$, $y = \sqrt{3}x$, $z = 1$
23. $G(x, y, z) = 24\sqrt{y}z$; S that portion of the cylinder $y = x^2$ in the first octant bounded by $y = 0$, $y = 4$, $z = 0$, $z = 3$
24. $G(x, y, z) = (1 + 4y^2 + 4z^2)^{1/2}$; S that portion of the paraboloid $x = 4 - y^2 - z^2$ in the first octant outside the cylinder $y^2 + z^2 = 1$

In Problems 25 and 26, evaluate $\iint_S (3z^2 + 4yz)\, dS$, where S is that portion of the plane $x + 2y + 3z = 6$ in the first octant. Use the projection of S onto the coordinate plane indicated in the given figure.

25.

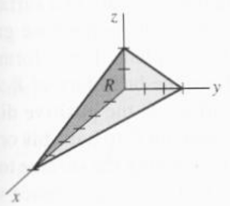

FIGURE 9.13.12 Region R for Problem 25

26.

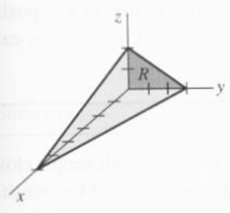

FIGURE 9.13.13 Region R for Problem 26

In Problems 27 and 28, find the mass of the given surface with the indicated density function.

27. S that portion of the plane $x + y + z = 1$ in the first octant; density at a point P directly proportional to the square of the distance from the yz-plane
28. S the hemisphere $z = \sqrt{4 - x^2 - y^2}$; $\rho(x, y, z) = |xy|$

In Problems 29–34, let $\mathbf{F}$ be a vector field. Find the flux of $\mathbf{F}$ through the given surface. Assume the surface S is oriented upward.

29. $\mathbf{F} = x\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}$; S that portion of the cylinder $y^2 + z^2 = 4$ in the first octant bounded by $x = 0$, $x = 3$, $y = 0$, $z = 0$
30. $\mathbf{F} = z\mathbf{k}$; S that part of the paraboloid $z = 5 - x^2 - y^2$ inside the cylinder $x^2 + y^2 = 4$
31. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; same surface S as in Problem 30
32. $\mathbf{F} = -x^3y\mathbf{i} + yz^3\mathbf{j} + xy^3\mathbf{k}$; S that portion of the plane $z = x + 3$ in the first octant within the cylinder $x^2 + y^2 = 2x$
33. $\mathbf{F} = \frac{1}{2}x^2\mathbf{i} + \frac{1}{2}y^2\mathbf{j} + z\mathbf{k}$; S that portion of the paraboloid $z = 4 - x^2 - y^2$ for $0 \le z \le 4$
34. $\mathbf{F} = e^y\mathbf{i} + e^x\mathbf{j} + 18y\mathbf{k}$; S that portion of the plane $x + y + z = 6$ in the first octant
35. Find the flux of $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j} + 5z\mathbf{k}$ out of the closed surface S given in Figure 9.13.11.
36. Find the flux of $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 6z^2\mathbf{k}$ out of the closed surface S bounded by the paraboloids $z = 4 - x^2 - y^2$ and $z = x^2 + y^2$.
37. Let $T(x, y, z) = x^2 + y^2 + z^2$ represent temperature and let the "flow" of heat be given by the vector field $\mathbf{F} = -\nabla T$. Find the flux of heat out of the sphere $x^2 + y^2 + z^2 = a^2$. [*Hint*: The surface area of a sphere of radius a is $4\pi a^2$.]
38. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ out of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. See **FIGURE 9.13.14**. Use the fact that the flux out of the cube is the sum of the fluxes out of the sides.

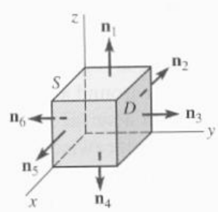

FIGURE 9.13.14 Cube in Problem 38

39. Coulomb's law states that the electric field $\mathbf{E}$ due to a point charge q at the origin is given by $\mathbf{E} = kq\mathbf{r}/\|\mathbf{r}\|^3$, where k is a constant and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Determine the flux out of a sphere $x^2 + y^2 + z^2 = a^2$.
40. If $\sigma(x, y, z)$ is charge density in an electrostatic field, then the total charge on a surface S is $Q = \iint_S \sigma(x, y, z)\, dS$. Find the total charge on that part of the hemisphere $z = \sqrt{16 - x^2 - y^2}$ that is inside the cylinder $x^2 + y^2 = 9$ if the charge density at a point P on the surface is directly proportional to distance from the xy-plane.
41. The coordinates of the centroid of a surface are given by

$$\bar{x} = \frac{\iint_S x\, dS}{A(S)}, \quad \bar{y} = \frac{\iint_S y\, dS}{A(S)}, \quad \bar{z} = \frac{\iint_S z\, dS}{A(S)},$$

where $A(S)$ is the area of the surface. Find the centroid of that portion of the plane $2x + 3y + z = 6$ in the first octant.

42. Use the information in Problem 41 to find the centroid of the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$.
43. Let $z = f(x, y)$ be the equation of a surface S and $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$. Show that $\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS$ equals

$$\iint_R \left[-P(x, y, z)\frac{\partial z}{\partial x} - Q(x, y, z)\frac{\partial z}{\partial y} + R(x, y, z)\right] dA.$$