Exercise 9.12

9.12 Exercises

Answers to selected odd-numbered problems begin on page ANS-22.

In Problems 1–4, verify Green's theorem by evaluating both integrals.

1. $\oint_C (x - y)\,dx + xy\,dy = \iint_R (y + 1)\,dA$, where C is the triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 3)$

2. $\oint_C 3x^2y\,dx + (x^2 - 5y)\,dy = \iint_R (2x - 3x^2)\,dA$, where C is the rectangle with vertices $(-1, 0)$, $(1, 0)$, $(1, 1)$, $(-1, 1)$

3. $\oint_C -y^2\,dx + x^2\,dy = \iint_R (2x + 2y)\,dA$, where C is the circle $x = 3\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$

4. $\oint_C -2y^2\,dx + 4xy\,dy = \iint_R 8y\,dA$, where C is the boundary of the region in the first quadrant determined by the graphs of $y = 0$, $y = \sqrt{x}$, $y = -x + 2$

In Problems 5–14, use Green's theorem to evaluate the given line integral.

5. $\oint_C 2y\,dx + 5x\,dy$, where C is the circle $(x - 1)^2 + (y + 3)^2 = 25$
6. $\oint_C (x + y^2)\,dx + (2x^2 - y)\,dy$, where C is the boundary of the region determined by the graphs of $y = x^2$, $y = 4$
7. $\oint_C (x^4 - 2y^3)\,dx + (2x^3 - y^4)\,dy$, where C is the circle $x^2 + y^2 = 4$
8. $\oint_C (x - 3y)\,dx + (4x + y)\,dy$, where C is the rectangle with vertices $(-2, 0)$, $(3, 0)$, $(3, 2)$, $(-2, 2)$
9. $\oint_C 2xy\,dx + 3xy^2\,dy$, where C is the triangle with vertices $(1, 2)$, $(2, 2)$, $(2, 4)$
10. $\oint_C e^{2x}\sin 2y\,dx + e^{2x}\cos 2y\,dy$, where C is the ellipse $9(x - 1)^2 + 4(y - 3)^2 = 36$
11. $\oint_C xy\,dx + x^2\,dy$, where C is the boundary of the region determined by the graphs of $x = 0$, $x^2 + y^2 = 1$, $x \ge 0$
12. $\oint_C e^{x^2}dx + 2\tan^{-1}x\,dy$, where C is the triangle with vertices $(0, 0)$, $(0, 1)$, $(-1, 1)$
13. $\oint_C \frac{1}{3}y^3\,dx + (xy + xy^2)\,dy$, where C is the boundary of the region in the first quadrant determined by the graphs of $y = 0$, $x = y^2$, $x = 1 - y^2$
14. $\oint_C xy^2\,dx + 3\cos y\,dy$, where C is the boundary of the region in the first quadrant determined by the graphs of $y = x^2$, $y = x^3$

In Problems 15 and 16, evaluate the given integral on any piecewise-smooth simple closed curve C.

15. $\oint_C ay\,dx + bx\,dy$
16. $\oint_C P(x)\,dx + Q(y)\,dy$

In Problems 17 and 18, let R be the region bounded by a piecewise-smooth simple closed curve C. Prove the given result.

17. $\oint_C x\,dy = -\oint_C y\,dx =$ area of R
18. $\frac{1}{2}\oint_C -y\,dx + x\,dy =$ area of R

In Problems 19 and 20, use the results of Problems 17 and 18 to find the area of the region bounded by the given closed curve.

19. The hypocycloid $x = a\cos^3 t$, $y = a\sin^3 t$, $a > 0$, $0 \le t \le 2\pi$
20. The ellipse $x = a\cos t$, $y = b\sin t$, $a > 0$, $b > 0$, $0 \le t \le 2\pi$
21. **(a)** Show that

$$\oint_C -y\,dx + x\,dy = x_1y_2 - x_2y_1,$$

where C is the line segment from the point (x_1, y_1) to (x_2, y_2).

(b) Use part (a) and Problem 18 to show that the area A of a polygon with vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, labeled counterclockwise, is

$$A = \frac{1}{2}(x_1y_2 - x_2y_1) + \frac{1}{2}(x_2y_3 - x_3y_2) + \cdots$$
$$+ \frac{1}{2}(x_{n-1}y_n - x_ny_{n-1}) + \frac{1}{2}(x_ny_1 - x_1y_n).$$

22. Use part (b) of Problem 21 to find the area of the quadrilateral

In Problems 23 and 24, evaluate the given line integral where $C = C_1 \cup C_2$ is the boundary of the shaded region R.

23. $\oint_C (4x^2 - y^3)\,dx + (x^3 + y^2)\,dy$

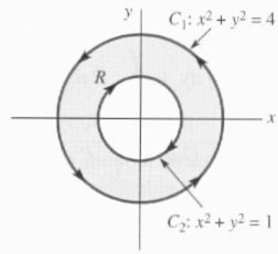

FIGURE 9.12.12 Boundary C for Problem 23

24. $\oint_C (\cos x^2 - y)\,dx + \sqrt{y^2 + 1}\,dy$

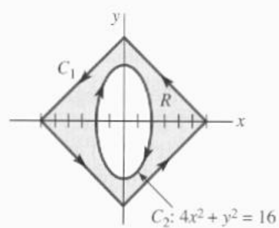

FIGURE 9.12.13 Boundary C for Problem 24

In Problems 25 and 26, proceed as in Example 6 to evaluate the given line integral.

25. $\displaystyle\oint_C \frac{-y^3dx + xy^2dy}{(x^2 + y^2)^2}$, where C is the ellipse $x^2 + 4y^2 = 4$

26. $\displaystyle\oint_C \frac{-y}{(x + 1)^2 + 4y^2}\,dx + \frac{x + 1}{(x + 1)^2 + 4y^2}\,dy$, where C is the circle $x^2 + y^2 = 16$

In Problems 27 and 28, use Green's theorem to evaluate the given double integral by means of a line integral. [*Hint*: Find appropriate functions P and Q.]

27. $\iint_R x^2\,dA$; R is the region bounded by the ellipse $x^2/9 + y^2/4 = 1$
28. $\iint_R [1 - 2(y - 1)]\,dA$; R is the region in the first quadrant bounded by the circle $x^2 + (y - 1)^2 = 1$ and $x = 0$

In Problems 29 and 30, use Green's theorem to find the work done by the given force $\mathbf{F}$ around the closed curve in **FIGURE 9.12.14**.

29. $\mathbf{F} = (x - y)\mathbf{i} + (x + y)\mathbf{j}$
30. $\mathbf{F} = -xy^2\mathbf{i} + x^2y\mathbf{j}$