Remarks

In everyday speech, the words *orthogonal*, *perpendicular*, and *normal* are often used interchangeably in the sense that two objects touch, intersect, or abut at a 90° angle. But in recent years an unwritten convention has arisen to use these terms in specific mathematical contexts. As a general rule, we say that two vectors are *orthogonal*, two lines (or two planes) are *perpendicular*, and that a vector is *normal* to a plane.

7.5 Exercises

Answers to selected odd-numbered problems begin on page ANS-14.

In Problems 1–6, find a vector equation for the line through the given points.

1. $(1, 2, 1), (3, 5, -2)$
2. $(0, 4, 5), (-2, 6, 3)$
3. $(\frac{1}{2}, -\frac{1}{2}, 1), (-\frac{3}{2}, \frac{5}{2}, -\frac{1}{2})$
4. $(10, 2, -10), (5, -3, 5)$
5. $(1, 1, -1), (-4, 1, -1)$
6. $(3, 2, 1), (\frac{5}{2}, 1, -2)$

In Problems 7–12, find parametric equations for the line through the given points.

7. $(2, 3, 5), (6, -1, 8)$
8. $(2, 0, 0), (0, 4, 9)$
9. $(1, 0, 0), (3, -2, -7)$
10. $(0, 0, 5), (-2, 4, 0)$
11. $(4, \frac{1}{2}, \frac{1}{3}), (-6, -\frac{1}{4}, \frac{1}{6})$
12. $(-3, 7, 9), (4, -8, -1)$

In Problems 13–18, find symmetric equations for the line through the given points.

13. $(1, 4, -9), (10, 14, -2)$
14. $(\frac{2}{3}, 0, -\frac{1}{4}), (1, 3, \frac{1}{4})$
15. $(4, 2, 1), (-7, 2, 5)$
16. $(-5, -2, -4), (1, 1, 2)$
17. $(5, 10, -2), (5, 1, -14)$
18. $(\frac{5}{6}, -\frac{1}{4}, \frac{1}{5}), (\frac{1}{3}, \frac{3}{8}, -\frac{1}{10})$

In Problems 19–22, find parametric and symmetric equations for the line through the given point parallel to the given vector.

19. $(4, 6, -7)$, $\mathbf{a} = \langle 3, \frac{1}{2}, -\frac{3}{2}\rangle$
20. $(1, 8, -2)$, $\mathbf{a} = -7\mathbf{i} - 8\mathbf{j}$
21. $(0, 0, 0)$, $\mathbf{a} = 5\mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$
22. $(0, -3, 10)$, $\mathbf{a} = \langle 12, -5, -6\rangle$
23. Find parametric equations for the line through $(6, 4, -2)$ that is parallel to the line $x/2 = (1 - y)/3 = (z - 5)/6$.
24. Find symmetric equations for the line through $(4, -11, -7)$ that is parallel to the line $x = 2 + 5t, y = -1 + \frac{1}{3}t, z = 9 - 2t$.
25. Find parametric equations for the line through $(2, -2, 15)$ that is parallel to the xz-plane and the xy-plane.
26. Find parametric equations for the line through $(1, 2, 8)$ that is **(a)** parallel to the y-axis, and **(b)** perpendicular to the xy-plane.
27. Show that the lines given by $\mathbf{r} = t\langle 1, 1, 1\rangle$ and $\mathbf{r} = \langle 6, 6, 6\rangle + t\langle -3, -3, -3\rangle$ are the same.
28. Let $\mathcal{L}_a$ and $\mathcal{L}_b$ be lines with direction vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. $\mathcal{L}_a$ and $\mathcal{L}_b$ are orthogonal if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal and parallel if $\mathbf{a}$ and $\mathbf{b}$ are parallel. Determine which of the following lines are orthogonal and which are parallel.
    (a) $\mathbf{r} = \langle 1, 0, 2\rangle + t\langle 9, -12, 6\rangle$
    (b) $x = 1 + 9t,\ y = 12t,\ z = 2 - 6t$
    (c) $x = 2t,\ y = -3t,\ z = 4t$
    (d) $x = 5 + t,\ y = 4t,\ z = 3 + \frac{5}{2}t$
    (e) $x = 1 + t,\ y = \frac{3}{2}t,\ z = 2 - \frac{3}{2}t$
    (f) $\dfrac{x + 1}{-3} = \dfrac{y + 6}{4} = \dfrac{z - 3}{-2}$

In Problems 29 and 30, determine the points of intersection of the given line and the three coordinate planes.

29. $x = 4 - 2t,\ y = 1 + 2t,\ z = 9 + 3t$
30. $\dfrac{x - 1}{2} = \dfrac{y + 2}{3} = \dfrac{z - 4}{2}$

In Problems 31–34, determine whether the given lines intersect. If so, find the point of intersection.

31. $x = 4 + t,\ y = 5 + t,\ z = -1 + 2t$
    $x = 6 + 2s,\ y = 11 + 4s,\ z = -3 + s$
32. $x = 1 + t,\ y = 2 - t,\ z = 3t$
    $x = 2 - s,\ y = 1 + s,\ z = 6s$
33. $x = 2 - t,\ y = 3 + t,\ z = 1 + t$
    $x = 4 + s,\ y = 1 + s,\ z = 1 - s$
34. $x = 3 - t,\ y = 2 + t,\ z = 8 + 2t$
    $x = 2 + 2s,\ y = -2 + 3s,\ z = -2 + 8s$

The angle between two lines $\mathcal{L}_a$ and $\mathcal{L}_b$ is the angle between their direction vectors $\mathbf{a}$ and $\mathbf{b}$. In Problems 35 and 36, find the angle between the given lines.

35. $x = 4 - t,\ y = 3 + 2t,\ z = -2t$
    $x = 5 + 2s,\ y = 1 + 3s,\ z = 5 - 6s$
36. $\dfrac{x - 1}{2} = \dfrac{y + 5}{7} = \dfrac{z - 1}{-1}$; $\dfrac{x + 3}{-2} = y - 9 = \dfrac{z}{4}$

In Problems 37 and 38, the given lines lie in the same plane. Find parametric equations for the line through the indicated point that is perpendicular to this plane.

37. $x = 3 + t,\ y = -2 + t,\ z = 9 + t$
    $x = 1 - 2s,\ y = 5 + s,\ z = -2 - 5s$; $(4, 1, 6)$
38. $\dfrac{x - 1}{3} = \dfrac{y + 1}{2} = \dfrac{z}{4}$
    $\dfrac{x + 4}{6} = \dfrac{y - 6}{4} = \dfrac{z - 10}{8}$; $(1, -1, 0)$

In Problems 39–44, find an equation of the plane that contains the given point and is perpendicular to the indicated vector.

39. $(5, 1, 3)$; $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$
40. $(1, 2, 5)$; $4\mathbf{i} - 2\mathbf{j}$
41. $(6, 10, -7)$; $-5\mathbf{i} + 3\mathbf{k}$
42. $(0, 0, 0)$; $6\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

43. $(\frac{1}{2}, \frac{3}{4}, -\frac{1}{2})$; $6\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$
44. $(-1, 1, 0)$; $-\mathbf{i} + \mathbf{j} - \mathbf{k}$

In Problems 45–50, find, if possible, an equation of a plane that contains the given points.

45. $(3, 5, 2)$, $(2, 3, 1)$, $(-1, -1, 4)$
46. $(0, 1, 0)$, $(0, 1, 1)$, $(1, 3, -1)$
47. $(0, 0, 0)$, $(1, 1, 1)$, $(3, 2, -1)$
48. $(0, 0, 3)$, $(0, -1, 0)$, $(0, 0, 6)$
49. $(1, 2, -1)$, $(4, 3, 1)$, $(7, 4, 3)$
50. $(2, 1, 2)$, $(4, 1, 0)$, $(5, 0, -5)$

In Problems 51–60, find an equation of the plane that satisfies the given conditions.

51. Contains $(2, 3, -5)$ and is parallel to $x + y - 4z = 1$
52. Contains the origin and is parallel to $5x - y + z = 6$
53. Contains $(3, 6, 12)$ and is parallel to the xy-plane
54. Contains $(-7, -5, 18)$ and is perpendicular to the y-axis
55. Contains the lines $x = 1 + 3t$, $y = 1 - t$, $z = 2 + t$; $x = 4 + 4s$, $y = 2s$, $z = 3 + s$
56. Contains the lines $\dfrac{x-1}{2} = \dfrac{y+1}{-1} = \dfrac{z-5}{6}$; $\mathbf{r} = \langle 1, -1, 5\rangle + t\langle 1, 1, -3\rangle$
57. Contains the parallel lines $x = 1 + t$, $y = 1 + 2t$, $z = 3 + t$; $x = 3 + s$, $y = 2s$, $z = -2 + s$
58. Contains the point $(4, 0, -6)$ and the line $x = 3t$, $y = 2t$, $z = -2t$
59. Contains $(2, 4, 8)$ and is perpendicular to the line $x = 10 - 3t$, $y = 5 + t$, $z = 6 - \frac{1}{2}t$
60. Contains $(1, 1, 1)$ and is perpendicular to the line through $(2, 6, -3)$ and $(1, 0, -2)$
61. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively. $\mathcal{P}_1$ and $\mathcal{P}_2$ are orthogonal if $\mathbf{n}_1$ and $\mathbf{n}_2$ are orthogonal and parallel if $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel. Determine which of the following planes are orthogonal and which are parallel.
    (a) $2x - y + 3z = 1$ (b) $x + 2y + 2z = 9$
    (c) $x + y - \frac{3}{2}z = 2$ (d) $-5x + 2y + 4z = 0$
    (e) $-8x - 8y + 12z = 1$ (f) $-2x + y - 3z = 5$
62. Find parametric equations for the line that contains $(-4, 1, 7)$ and is perpendicular to the plane $-7x + 2y + 3z = 1$.
63. Determine which of the following planes are perpendicular to the line $x = 4 - 6t$, $y = 1 + 9t$, $z = 2 + 3t$.
    (a) $4x + y + 2z = 1$ (b) $2x - 3y + z = 4$
    (c) $10x - 15y - 5z = 2$ (d) $-4x + 6y + 2z = 9$
64. Determine which of the following planes are parallel to the line $(1 - x)/2 = (y + 2)/4 = z - 5$.
    (a) $x - y + 3z = 1$ (b) $6x - 3y = 1$
    (c) $x - 2y + 5z = 0$ (d) $-2x + y - 2z = 7$

In Problems 65–68, find parametric equations for the line of intersection of the given planes.

65. $5x - 4y - 9z = 8$, $x + 4y + 3z = 4$
66. $x + 2y - z = 2$, $3x - y + 2z = 1$
67. $4x - 2y - z = 1$, $x + y + 2z = 1$
68. $2x - 5y + z = 0$, $y = 0$

In Problems 69–72, find the point of intersection of the given plane and line.

69. $2x - 3y + 2z = -7$; $x = 1 + 2t$, $y = 2 - t$, $z = -3t$
70. $x + y + 4z = 12$; $x = 3 - 2t$, $y = 1 + 6t$, $z = 2 - \frac{1}{2}t$
71. $x + y - z = 8$; $x = 1$, $y = 2$, $z = 1 + t$
72. $x - 3y + 2z = 0$; $x = 4 + t$, $y = 2 + t$, $z = 1 + 5t$

In Problems 73 and 74, find parametric equations for the line through the indicated point that is parallel to the given planes.

73. $x + y - 4z = 2$, $2x - y + z = 10$; $(5, 6, -12)$
74. $2x + z = 0$, $-x + 3y + z = 1$; $(-3, 5, -1)$

In Problems 75 and 76, find an equation of the plane that contains the given line and is orthogonal to the indicated plane.

75. $x = 4 + 3t$, $y = -t$, $z = 1 + 5t$; $x + y + z = 7$
76. $\dfrac{2-x}{3} = \dfrac{y+2}{5} = \dfrac{z-8}{2}$; $2x - 4y - z + 16 = 0$

In Problems 77–82, graph the given equation.

77. $5x + 2y + z = 10$
78. $3x + 2z = 9$
79. $-y - 3z + 6 = 0$
80. $3x + 4y - 2z - 12 = 0$
81. $-x + 2y + z = 4$
82. $x - y - 1 = 0$

7.6 Vector Spaces

Introduction In the preceding sections we were dealing with points and vectors in 2- and 3-space. Mathematicians in the nineteenth century, notably the English mathematicians Arthur Cayley (1821–1895) and James Joseph Sylvester (1814–1897) and the Irish mathematician William Rowan Hamilton (1805–1865), realized that the concepts of *point* and *vector* could be generalized. A realization developed that vectors could be described, or defined, by analytic rather than geometric properties. This was a truly significant breakthrough in the history of mathematics. There is no need to stop with three dimensions; ordered quadruples $\langle a_1, a_2, a_3, a_4\rangle$, quintuples $\langle a_1, a_2, a_3, a_4, a_5\rangle$, and n-tuples $\langle a_1, a_2, \ldots, a_n\rangle$ of real numbers can be thought of as vectors just as well as ordered pairs $\langle a_1, a_2\rangle$ and ordered triples $\langle a_1, a_2, a_3\rangle$; the only difference being that we lose our ability to visualize directed line segments or arrows in 4-dimensional, 5-dimensional, or n-dimensional space.