5.4 Periodic noise reduction by frequency domain filtering (page 357)

5.4.1 Bandreject Filters

Ideal bandreject filter

$$H(u,v) = \begin{cases} 1, & D(u,v) < D_0 - \frac{W}{2} \\ 0, & D_0 - \frac{W}{2} \le D(u,v) \le D_0 + \frac{W}{2} \\ 1, & D(u,v) > D_0 + \frac{W}{2} \end{cases}$$

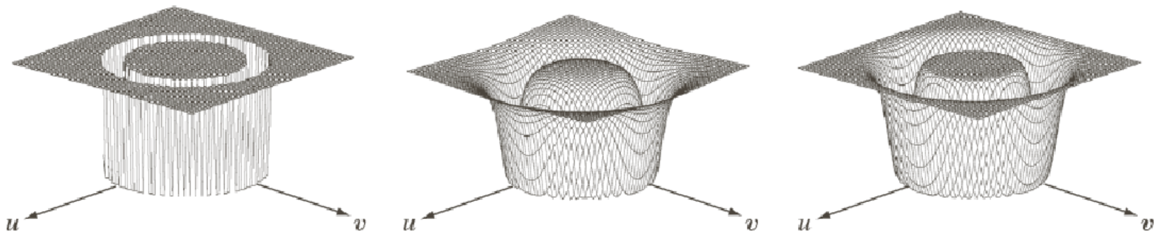
W **is the width of the band**

Butterworth bandreject filter

$$H(u,v) = \frac{1}{1 + \left\{ \dfrac{D(u,v)\, W}{D^2(u,v) - D_0^2} \right\}^{2n}}$$

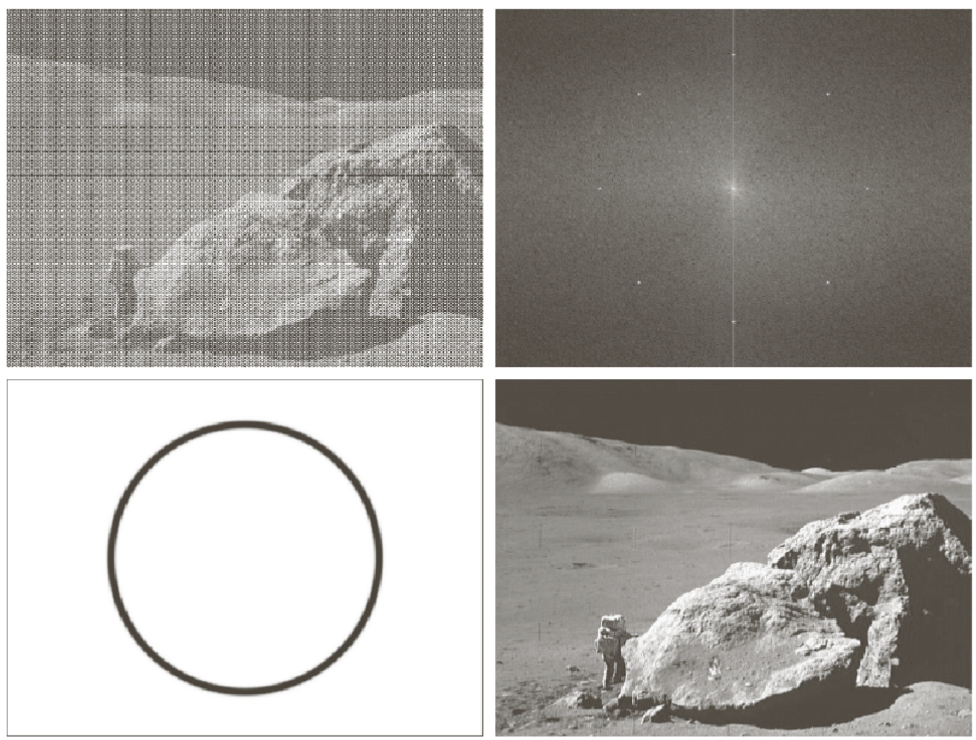
Gaussian bandreject filter

$$H(u,v) = 1 - e^{-\left\{ \frac{D^2(u,v) - D_0^2}{D(u,v)\, W} \right\}}$$



a b c

FIGURE 5.15 From left to right, perspective plots of ideal, Butterworth (of order 1), and Gaussian bandreject filters.



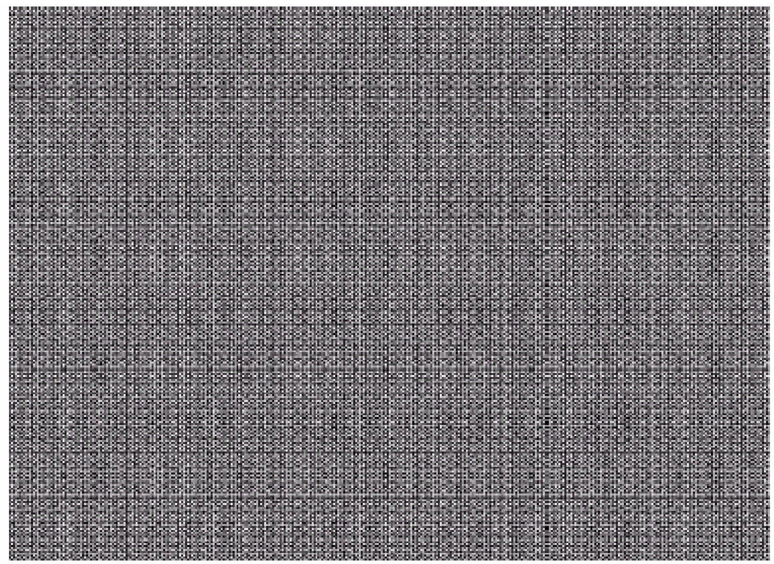
a b
c d

FIGURE 5.16 (a) Image corrupted by sinusoidal noise. (b) Spectrum of (a). (c) Butterworth bandreject filter (white represents 1). (d) Result of filtering. (Original image courtesy of NASA.)

5.4.2 Bandpass Filters

$$H_{\mathrm{BP}}(u,v) = 1 - H_{\mathrm{BR}}(u,v)$$

FIGURE 5.17 Noise pattern of the image in Fig. 5.16(a) obtained by bandpass filtering.



5.4.3 Notch Filters

Ideal notch reject filter

$$H(u,v) = \begin{cases} 0, & D_1(u,v) \le D_0 \textbf{ or } D_2(u,v) \le D_0 \\ 1, & \textbf{otherwise} \end{cases}$$

$$D_1(u,v) = \left[\, (u - M/2 - u_0)^2 + (v - N/2 - v_0)^2 \,\right]^{1/2}$$
$$D_2(u,v) = \left[\, (u - M/2 + u_0)^2 + (v - N/2 + v_0)^2 \,\right]^{1/2}$$

Butterworth notch reject filter

$$H(u,v) = \frac{1}{1 + \left\{ \dfrac{D_0^2}{D_1(u,v)\, D_2(u,v)} \right\}^{2n}}$$

Gaussian notch reject filter

$$H(u,v) = 1 - e^{-\left\{ \frac{D_1(u,v)\, D_2(u,v)}{D_0^2} \right\}}$$

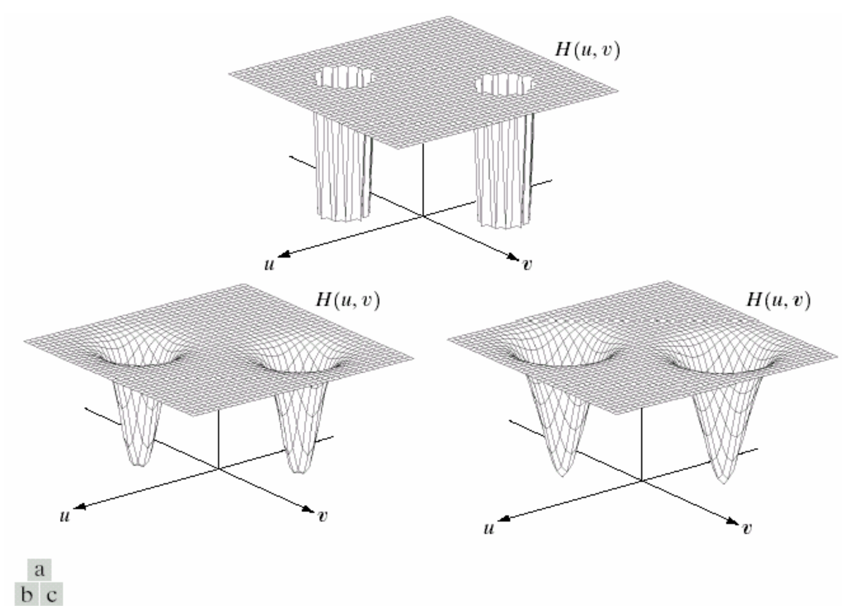
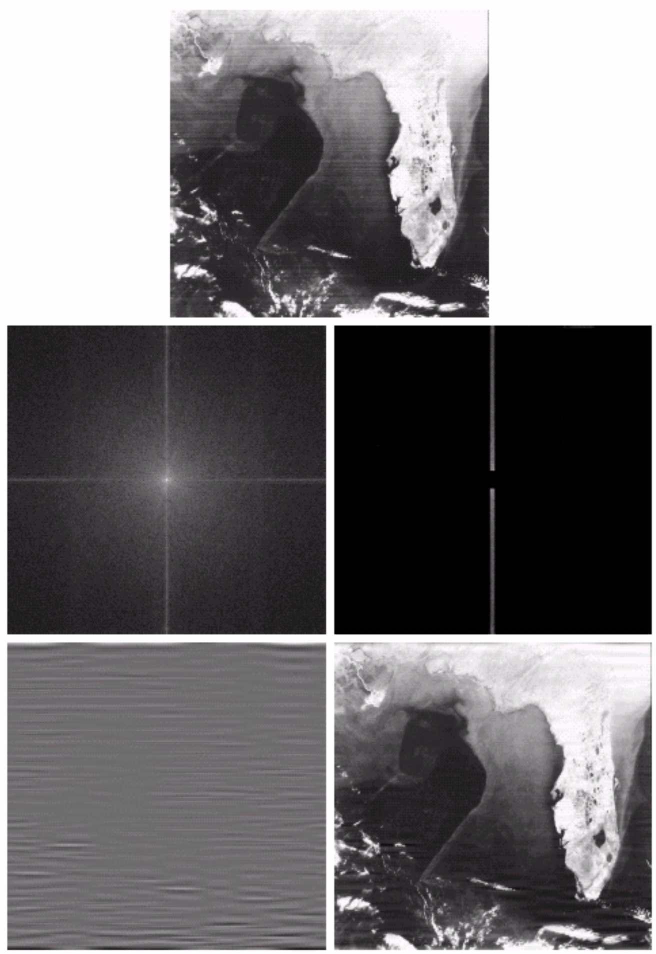


FIGURE 5.18 Perspective plots of (a) ideal, (b) Butterworth (of order 2), and (c) Gaussian notch (reject) filters.

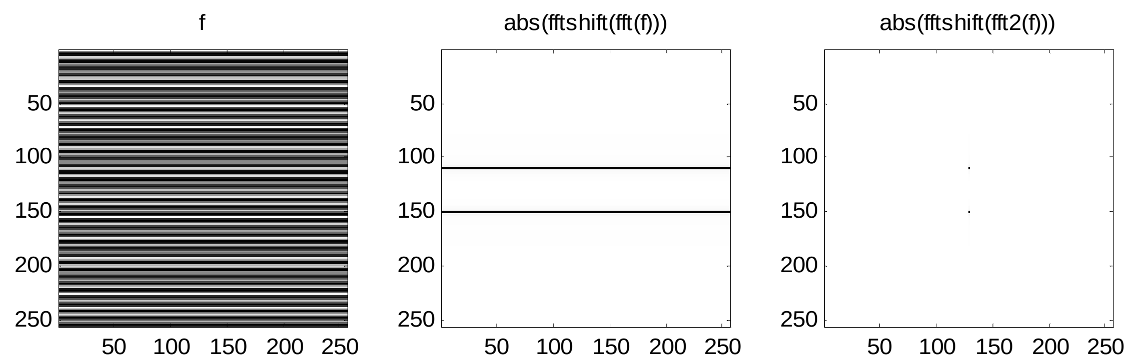
Notch pass filters

$$H_{\mathrm{NP}}(u,v) = 1 - H_{\mathrm{NR}}(u,v)$$

Example 5.8: Removal of periodic noise by notch filtering



Explanation of Example 5.8:



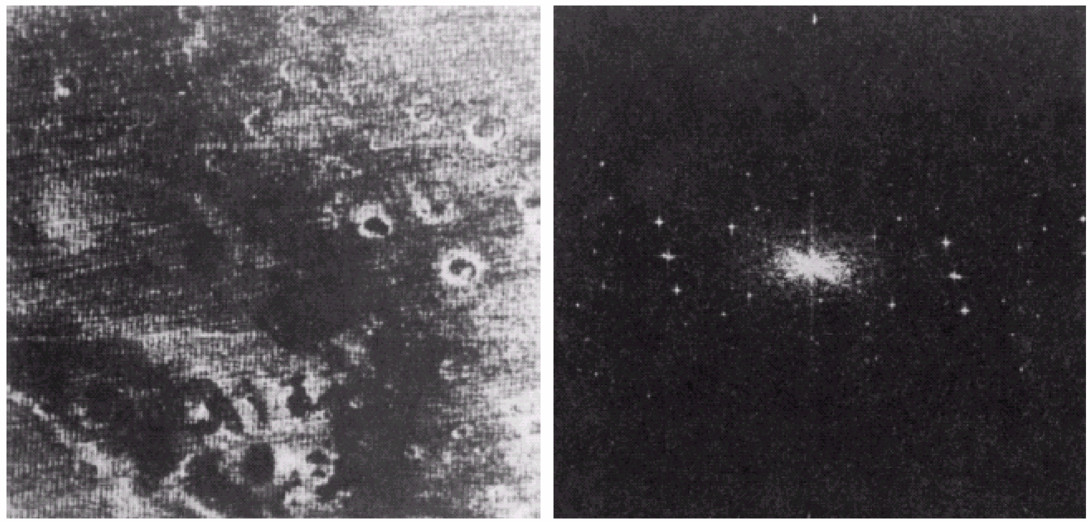
Each of the columns of the image above left contains a discretization `f` **of the function** $f(x) = \sin(20x)\ x \in [0, 2\pi]$**. The negative of the image is displayed.**

The image in the middle shows the negative of `abs( fftshift( fft (f) ) )`

The image above right shows the negative of `abs( fftshift( fft2 (f) ) )`

5.4.4 Optimum Notch Filtering

Starlike components in Fourier spectrum indicate more than one sinusoidal pattern



a b

FIGURE 5.20 (a) Image of the Martian terrain taken by *Mariner 6*. (b) Fourier spectrum showing periodic interference. (Courtesy of NASA.)

Observe $G(u,v)$ and experiment with different notch pass filters $H_{\mathrm{NP}}(u,v)$ where

$$\eta(x,y) = \textbf{IFT}\left\{ H_{\mathrm{NP}}(u,v)\, G(u,v) \right\}$$

We want to optimize a weighting or modulation function $w(x,y)$ where

$$\hat{f}(x,y) = g(x,y) - w(x,y)\,\eta(x,y) \tag{1}$$

in such a way that local variances of $\hat{f}(x,y)$ is minimized

Consider a neighbourhood size of $(2a+1)$ by $(2b+1)$ about every point

Local variance of $\hat{f}(x,y)$ at (x,y):

$$\sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} \left[\, \hat{f}(x+s,y+t) - \overline{\hat{f}}(x,y) \,\right]^2 \tag{2}$$

$$\overline{\hat{f}}(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} \hat{f}(x+s,y+t)$$

Substitute (1) into (2):

$$\begin{aligned} \sigma^2(x,y) \;=\; & \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} \Big\{ \big[\, g(x+s,y+t) \\ & - w(x+s,y+t)\, \eta(x+s,y+t) \,\big] \\ & - \big[\, \overline{g}(x,y) - \overline{w(x,y)\, \eta(x,y)} \,\big] \Big\}^2 \end{aligned}$$

Assume that $w(x,y)$ remains constant over the neighbourhood, that is let

$$w(x+s,y+t) = w(x,y),$$

then

$$\overline{w(x,y)\, \eta(x,y)} = w(x,y)\, \overline{\eta}(x,y)$$

and

$$\sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} \{ [g(x+s,y+t) - w(x,y)\,\eta(x+s,y+t)] - [\overline{g}(x,y) - w(x,y)\,\overline{\eta}(x,y)] \}^2$$

To minimize $\sigma^2(x,y)$**, we solve** $\dfrac{\partial\,\sigma^2(x,y)}{\partial\,w(x,y)} = 0$ **for** $w(x,y)$**.**

The result is

$$w(x,y) = \frac{\overline{g(x,y)\,\eta(x,y)} - \overline{g}(x,y)\,\overline{\eta}(x,y)}{\overline{\eta^2(x,y)} - [\overline{\eta}(x,y)]^2}$$

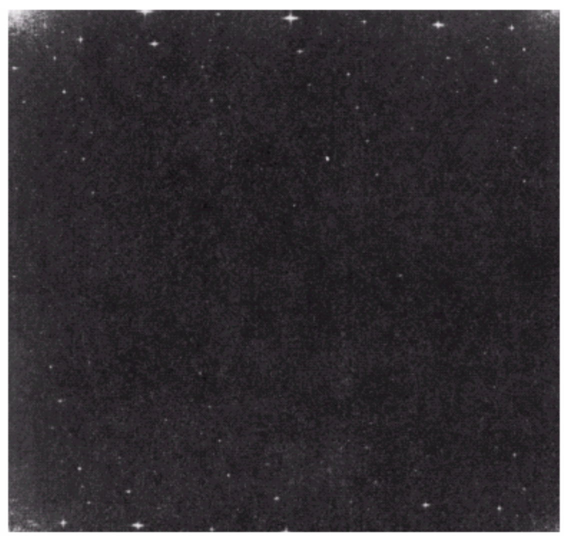
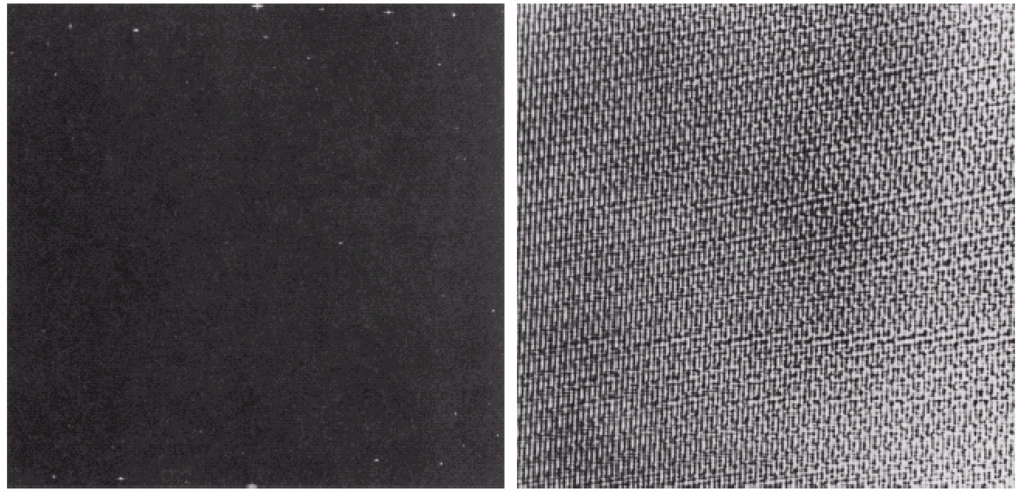


FIGURE 5.21 Fourier spectrum (without shifting) of the image shown in Fig. 5.20(a). (Courtesy of NASA.)



a b

FIGURE 5.22 (a) Fourier spectrum of $N(u, v)$, and (b) corresponding noise interference pattern $\eta(x, y)$. (Courtesy of NASA.)

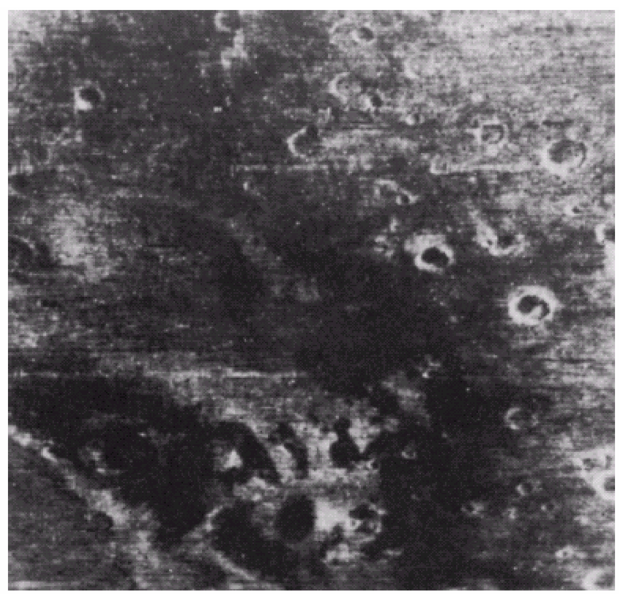


FIGURE 5.23 Processed image. (Courtesy of NASA.)

5.5 Linear, position-invariant degradations

$$
\begin{aligned}
g(x,y) &= (h * f)(x,y) + \eta(x,y) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\, h(x-\alpha, y-\beta)\, d\alpha\, d\beta + \eta(x,y)
\end{aligned}
$$

$$G(u,v) = H(u,v)\, F(u,v) + N(u,v)$$

- **Image deconvolution**
- **Deconvolution filters**

5.6 Estimating the degradation function

(1) Observation

(2) Experimentation

(3) Mathematical modelling

5.6.1 Estimation by image observation

- **Given: only the degraded image**
- **Choose subimage that contains simple structures with little noise:** $g_s(x,y)$

- **Estimate original subimage:** $\hat{f}_s(x,y)$

$$H_s(u,v) = \frac{G_s(u,v)}{\hat{F}_s(u,v)}$$

- **Construct** $H(u,v)$ **on a larger scale with similar "shape" as** $H_s(u,v)$

5.6.2 Estimation by experimentation

- **Given: degraded image AND similar acquisition equipment**
- **Change settings until image resembles degraded image**
- **Aqcuire image of impulse (dot of light) with same settings**

$$H(u,v) = \frac{G(u,v)}{A}, \text{ with } A \text{ the strength of impulse}$$

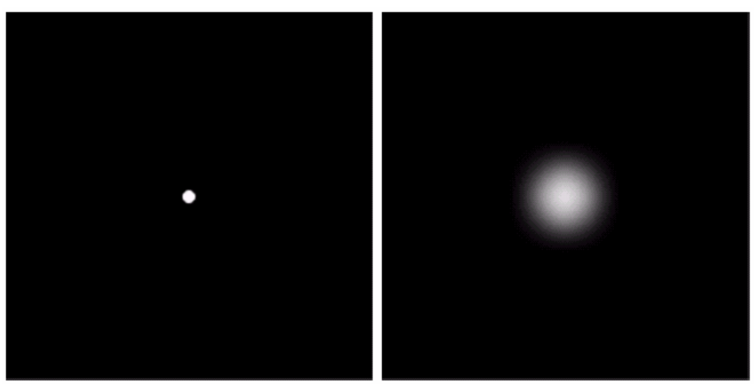


FIGURE 5.24 Degradation estimation by impulse characterization. (a) An impulse of light (shown magnified). (b) Imaged (degraded) impulse.

5.6.3 Estimation by modelling

- **Atmospheric turbulence**

$$H(u,v) = e^{-k(u^2+v^2)^{5/6}}$$

a b
c d

FIGURE 5.25 Illustration of the atmospheric turbulence model. (a) Negligible turbulence. (b) Severe turbulence, $k = 0.0025$. (c) Mild turbulence, $k = 0.001$. (d) Low turbulence, $k = 0.00025$. (Original image courtesy of NASA.)

- **Basic mathematical principles: uniform linear motion**

Let T **be duration of exposure, then** $g(x,y) = \displaystyle\int_0^T f[\,x - x_0(t),\; y - y_0(t)\,]\,dt$

$$\begin{aligned}
G(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, e^{-2\pi i(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\int_0^T f[\,x - x_0(t),\; y - y_0(t)\,]\,dt\right) e^{-2\pi i(ux+vy)}\,dx\,dy \\
&= \int_0^T \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f[\,x - x_0(t),\; y - y_0(t)\,]\, e^{-2\pi i(ux+vy)}\,dx\,dy\right) dt \\
&= \int_0^T F(u,v)\, e^{-2\pi i[ux_0(t)+vy_0(t)]}\,dt \\
&= F(u,v) \int_0^T e^{-2\pi i[ux_0(t)+vy_0(t)]}\,dt
\end{aligned}$$

Define $H(u,v) = \displaystyle\int_0^T e^{-2\pi i[ux_0(t)+vy_0(t)]}\,dt$ **so that** $G(u,v) = H(u,v)\,F(u,v)$

If $x_0(t)$ **and** $y_0(t)$ **are known, then** $H(u,v)$ **is known**

Illustration

Let $x_0(t) = \dfrac{at}{T}$ **and** $y_0(t) = 0$

Note that $x_0(T) = a$

$$\begin{aligned} H(u,v) &= \int_0^T e^{-2\pi i u x_0(t)}\,dt \\ &= \int_0^T e^{-2\pi i u a t/T}\,dt \\ &= \vdots \\ &= \frac{T}{\pi u a}\ \sin(\pi u a)\ e^{-\pi i u a} \end{aligned}$$

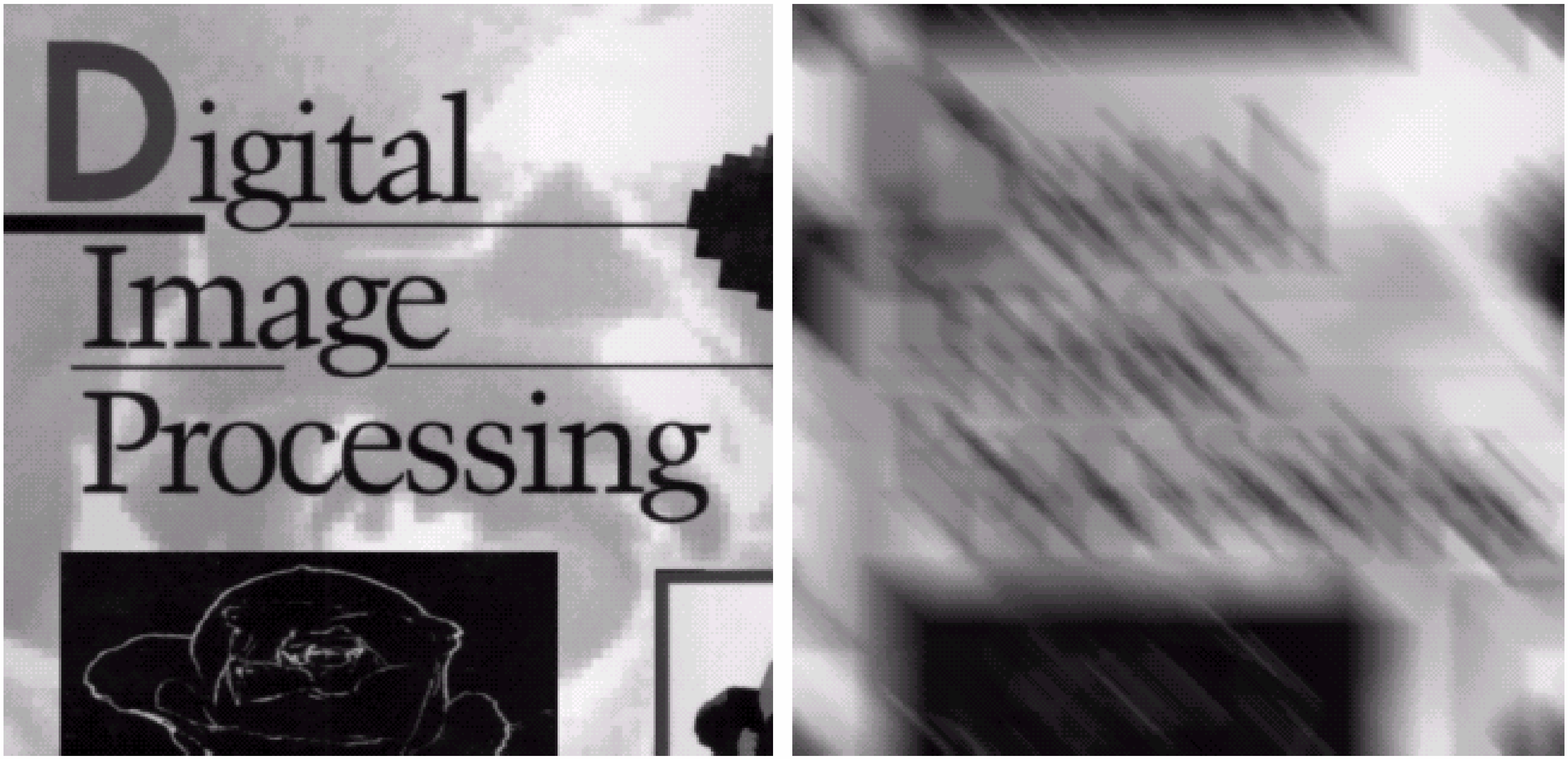
(verify)

Note that $H(u,v) = 0$ **if** $u = \dfrac{n}{a},\ n \in Z$

When $x_0(t) = \dfrac{at}{T}$ **and** $y_0(t) = \dfrac{bt}{T}$**, then**

$$H(u,v) = \frac{T}{\pi(ua+vb)}\ \sin[\pi(ua+vb)]\ e^{-\pi i(ua+vb)}$$

Example 5.10: Image blurring due to motion



a b

FIGURE 5.26 (a) Original image. (b) Result of blurring using the function in Eq. (5.6-11) with $a = b = 0.1$ and $T = 1$.