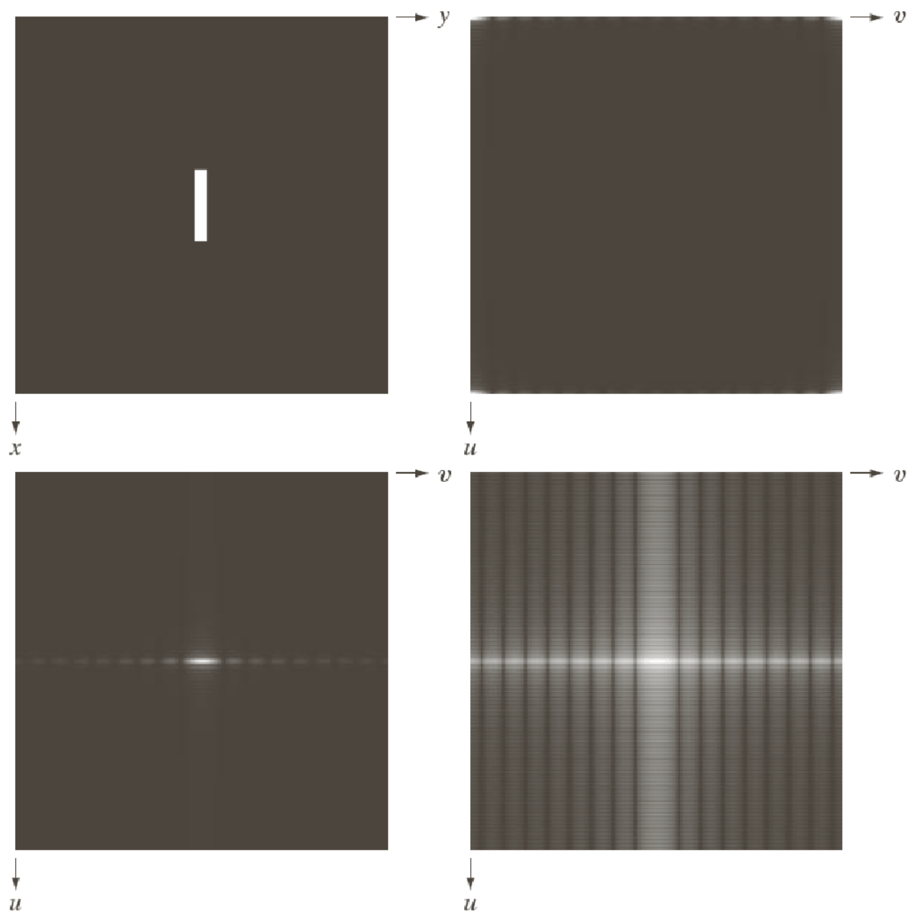
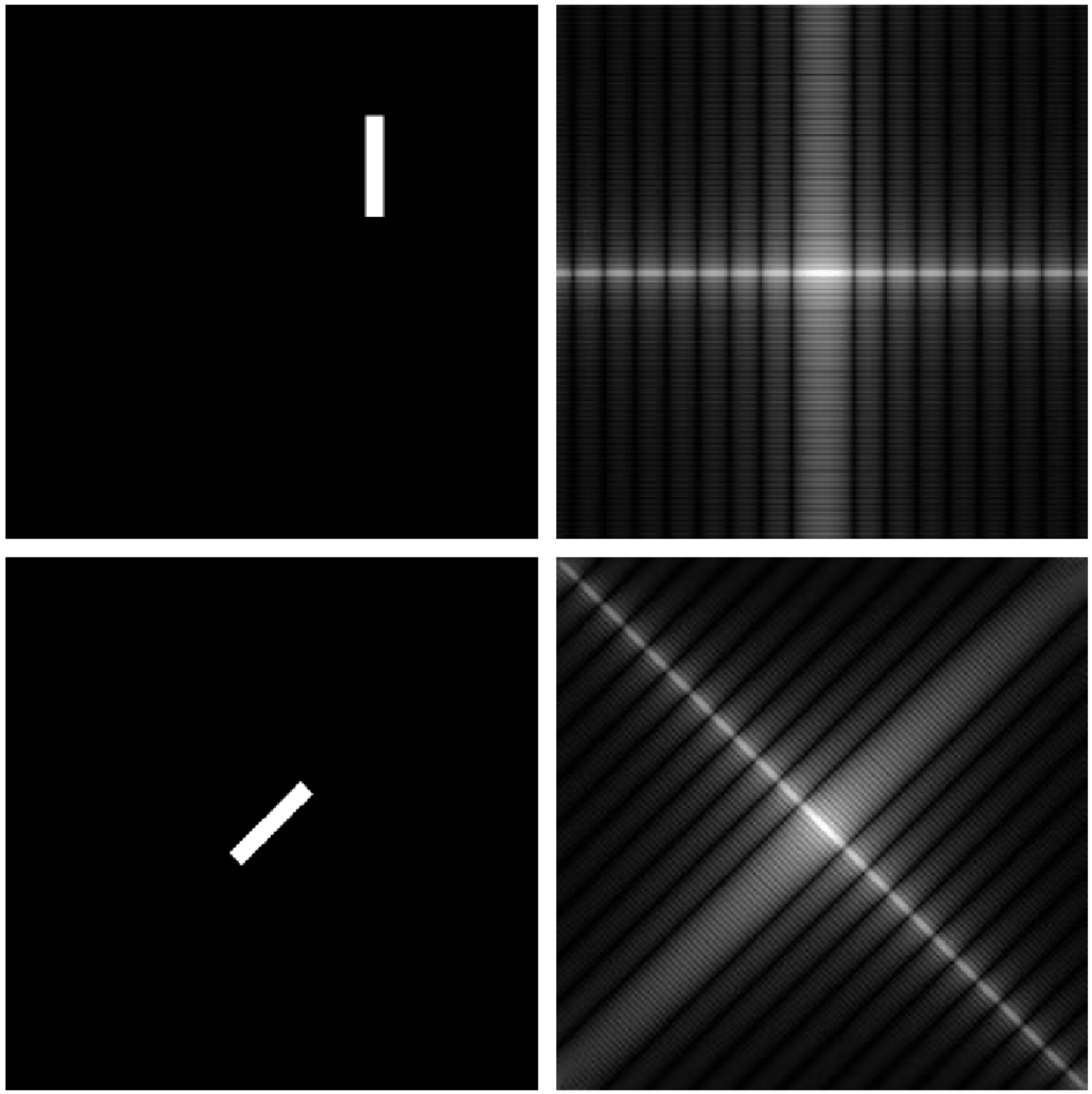
4.7 The basics of filtering in the frequency domain (page 277)

Example: An image and its Fourier spectrum

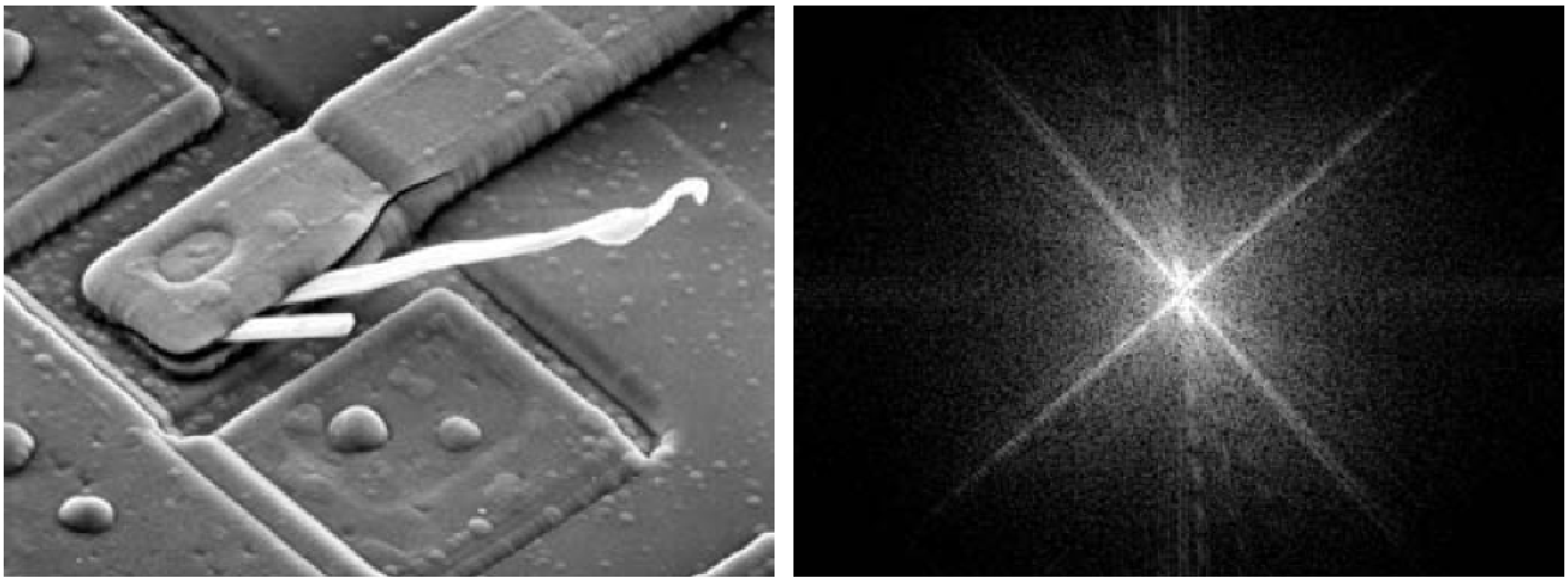


Left-to-right and top-to-bottom: (1) `f(x,y)`, **(2)** `abs(fft2(f(x,y)))`, **(3)** `abs(fftshift(fft2(f(x,y))))`, **(4) log transformation of (3)**

Example: Two images and their Fourier spectra



Example: An image and its Fourier spectrum



a b

FIGURE 4.29 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

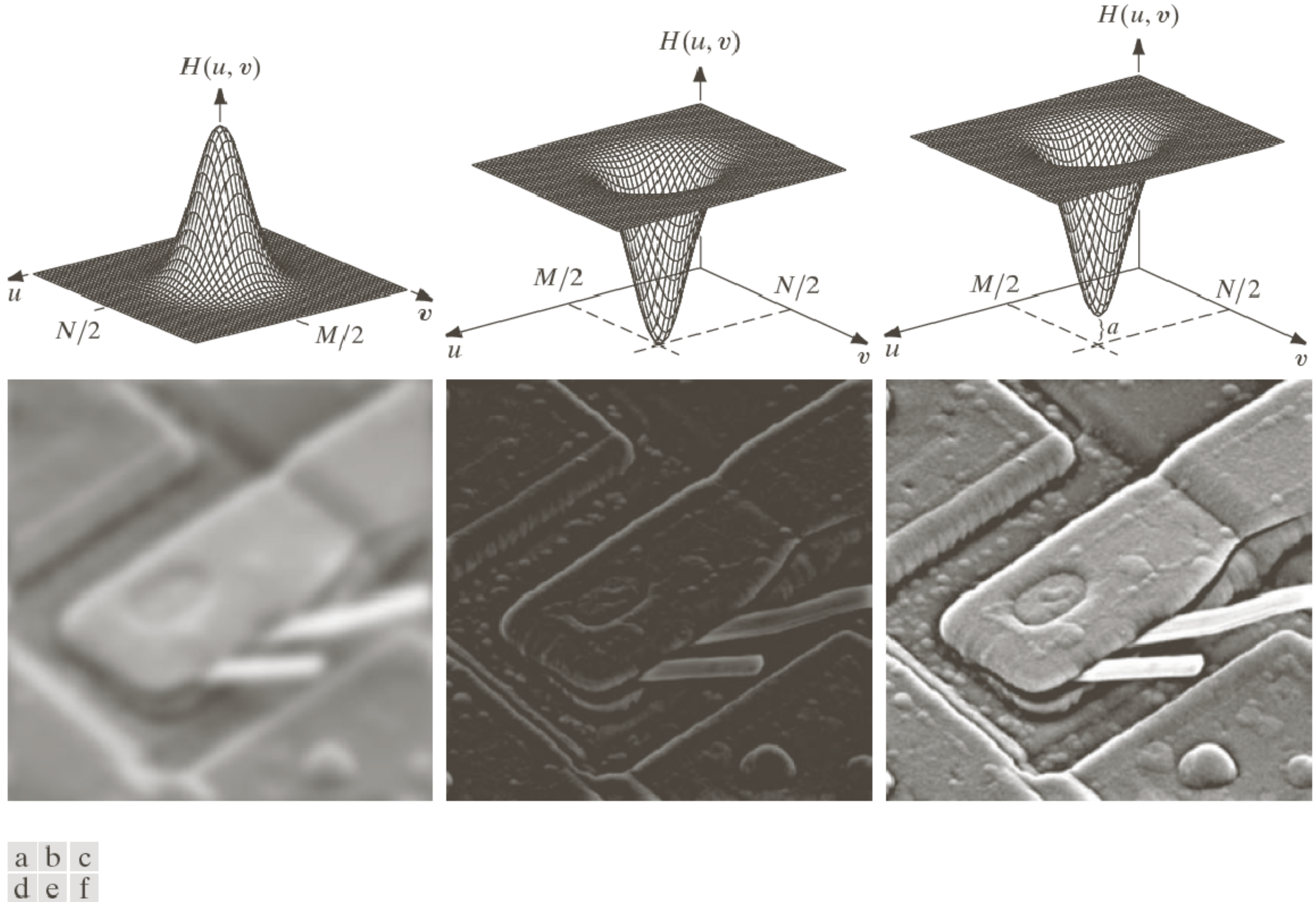


FIGURE 4.31 Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

From left to right: (a) Lowpass filter, (b) Highpass filter, (c) Small constant a added to (b)

Steps for filtering in the frequency domain

Input: $M \times N$ **image** $f(x, y)$ **Output:** $M \times N$ **image** $g(x, y)$

Padded images: $P \times Q$ **images** $f_P(x, y)$ **and** $g_P(x, y)$ **(**$P = 2M$**,** $Q = 2N$**)**

(1) **Zero pad** $f(x, y) \Rightarrow f_P(x, y)$

(2) **Multiply** $f_P(x, y)$ **by** $(-1)^{(x+y)}$

(3) **Compute** $F(u, v)$**, that is the FT of the result in (2)**

(4) **Multiply** $F(u, v)$ **by** $H(u, v)$ **(array multiplication)**

(5) **Compute the IFT of the result in (4)**

(6) **Obtain the real part of the result in (5)**

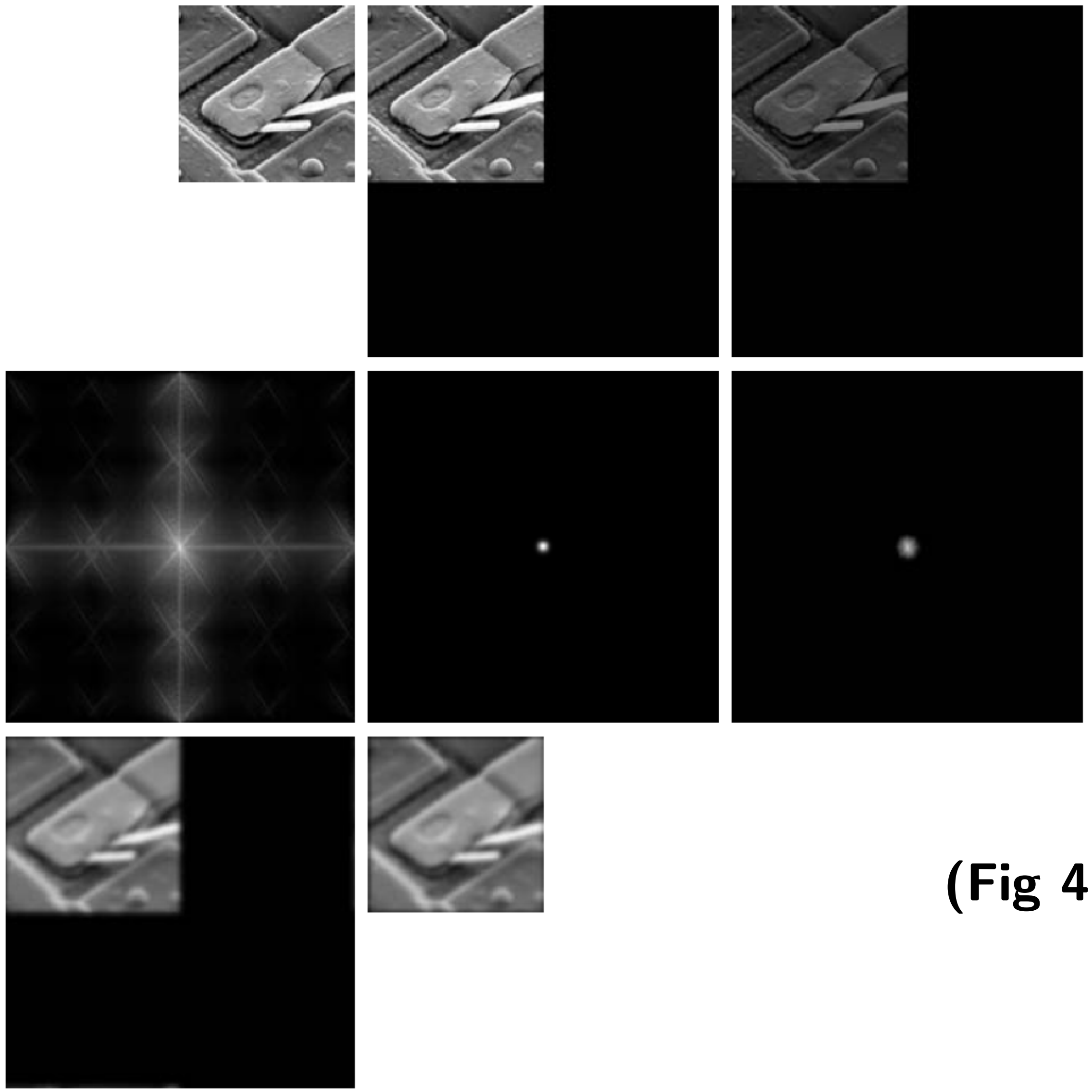
(7) **Multiply the result in (6) by** $(-1)^{(x+y)} \Rightarrow g_P(x, y)$

(8) **Crop** $g_P(x, y) \Rightarrow g(x, y)$

Matlab command, using the `fftshift` function

```
>> g_P = real(ifft2(fftshift((fftshift(fft2(f_P))).* H)));
```

Padding in the frequency domain eliminates wraparound error



(Fig 4.36: page 286)

Example of wraparound error

a b c

FIGURE 4.32 (a) A simple image. (b) Result of blurring with a Gaussian lowpass filter without padding. (c) Result of lowpass filtering with padding. Compare the light area of the vertical edges in (b) and (c).

Explanation of wraparound error

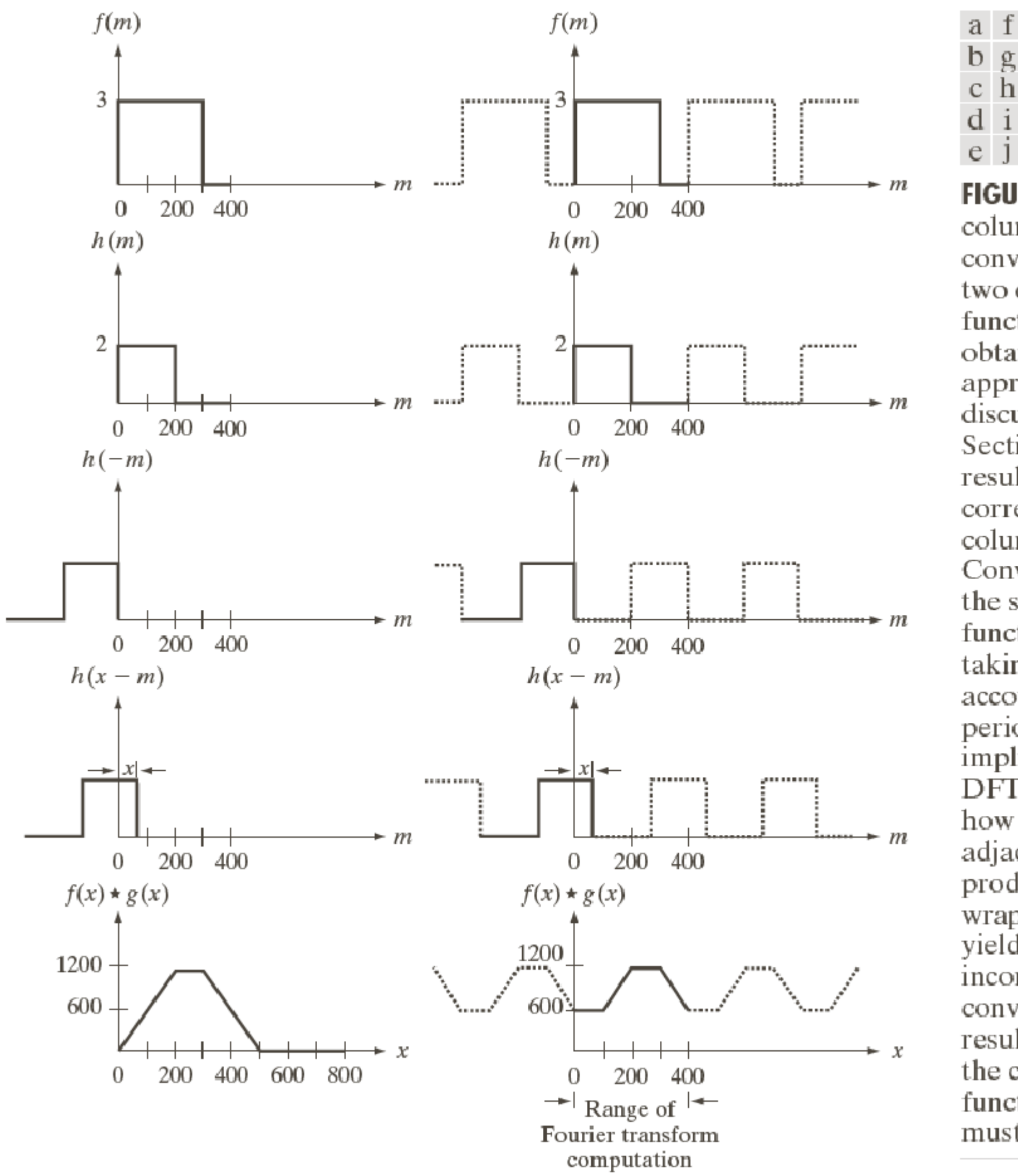


FIGURE 4.28 Left column: convolution of two discrete functions obtained using the approach discussed in Section 3.4.2. The result in (e) is correct. Right column: Convolution of the same functions, but taking into account the periodicity implied by the DFT. Note in (j) how data from adjacent periods produce wraparound error, yielding an incorrect convolution result. To obtain the correct result, function padding must be used.

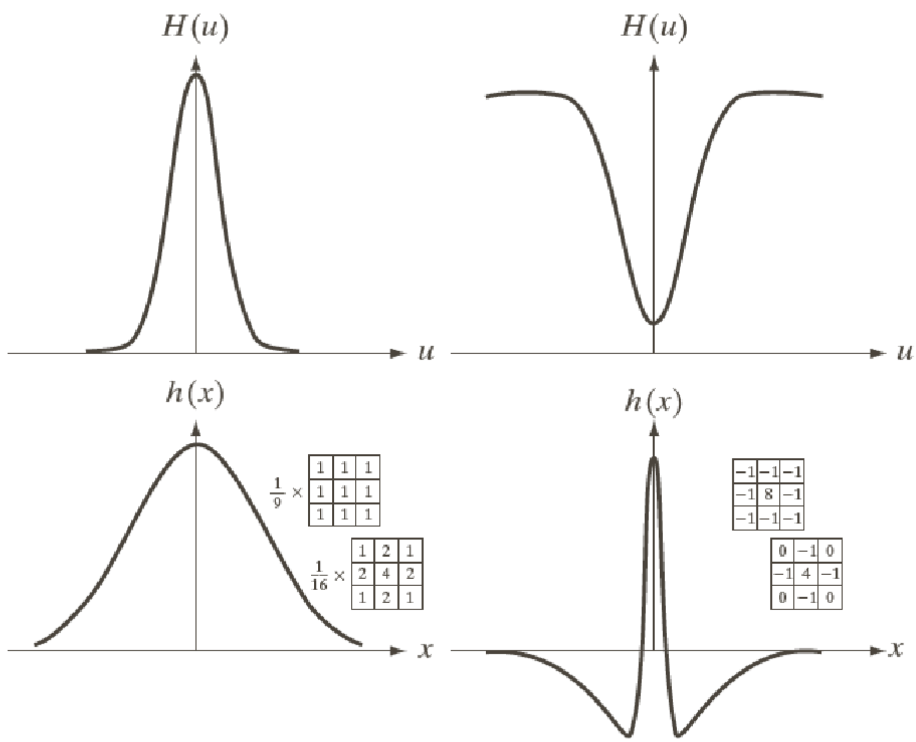
Smoothing spatial filter $\Leftrightarrow$ Lowpass filter in frequency space

$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2 x^2} \quad \Leftrightarrow \quad H(u) = Ae^{-u^2/2\sigma^2}$$

Sharpening spatial filter $\Leftrightarrow$ Highpass filter in frequency space

$$h(x) = \sqrt{2\pi}\sigma_1 A e^{-2\pi^2\sigma_1^2 x^2} - \sqrt{2\pi}\sigma_2 B e^{-2\pi^2\sigma_2^2 x^2} \quad \Leftrightarrow$$

$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}, \quad A \geq B,\ \sigma_1 > \sigma_2$$



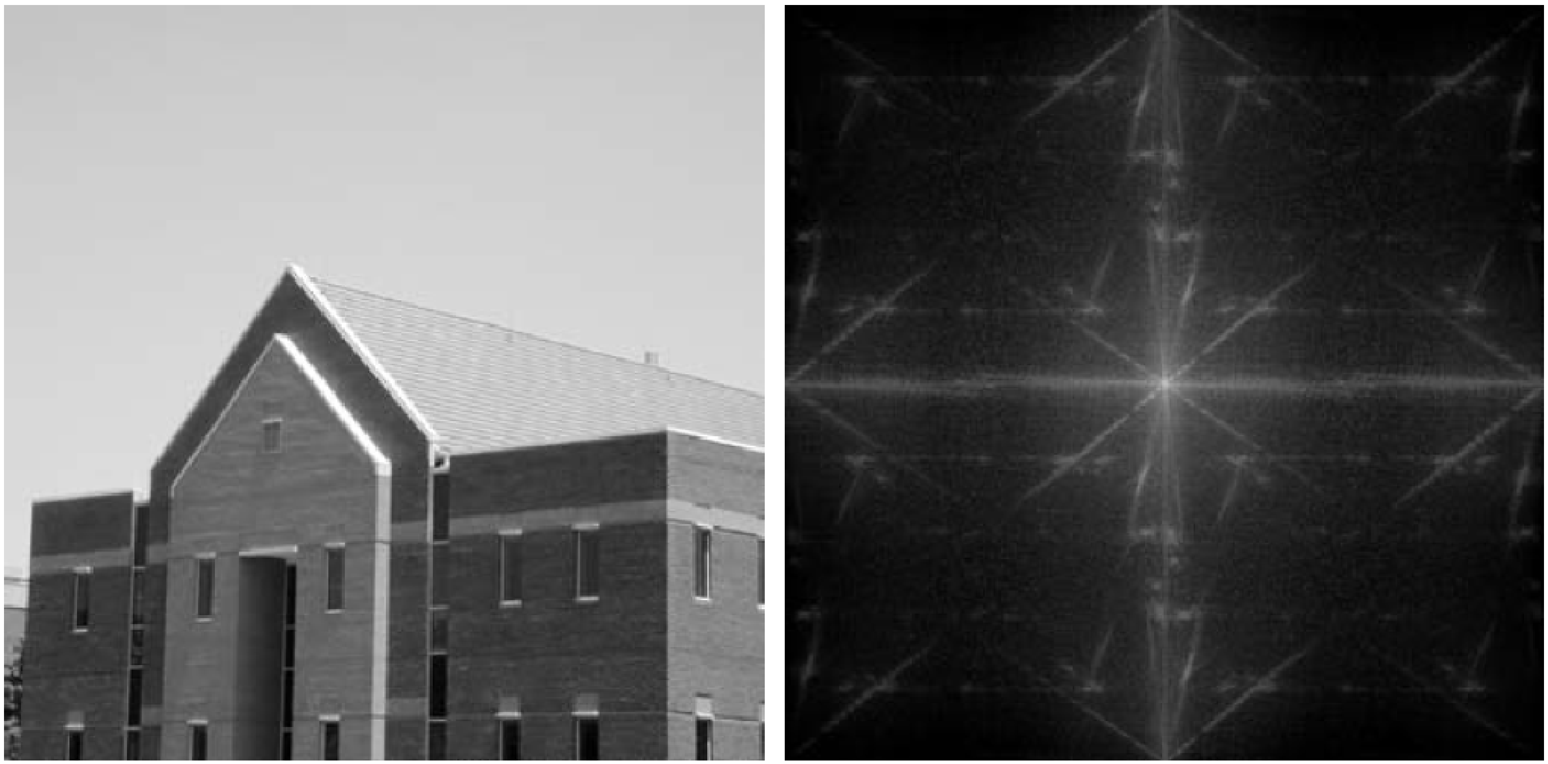
a c
b d

FIGURE 4.37 (a) A 1-D Gaussian lowpass filter in the frequency domain. (b) Spatial lowpass filter corresponding to (a). (c) Gaussian highpass filter in the frequency domain. (d) Spatial highpass filter corresponding to (c). The small 2-D masks shown are spatial filters we used in Chapter 3.

Example 4.15

Obtaining a frequency domain filter from a small spatial mask

(Self study)



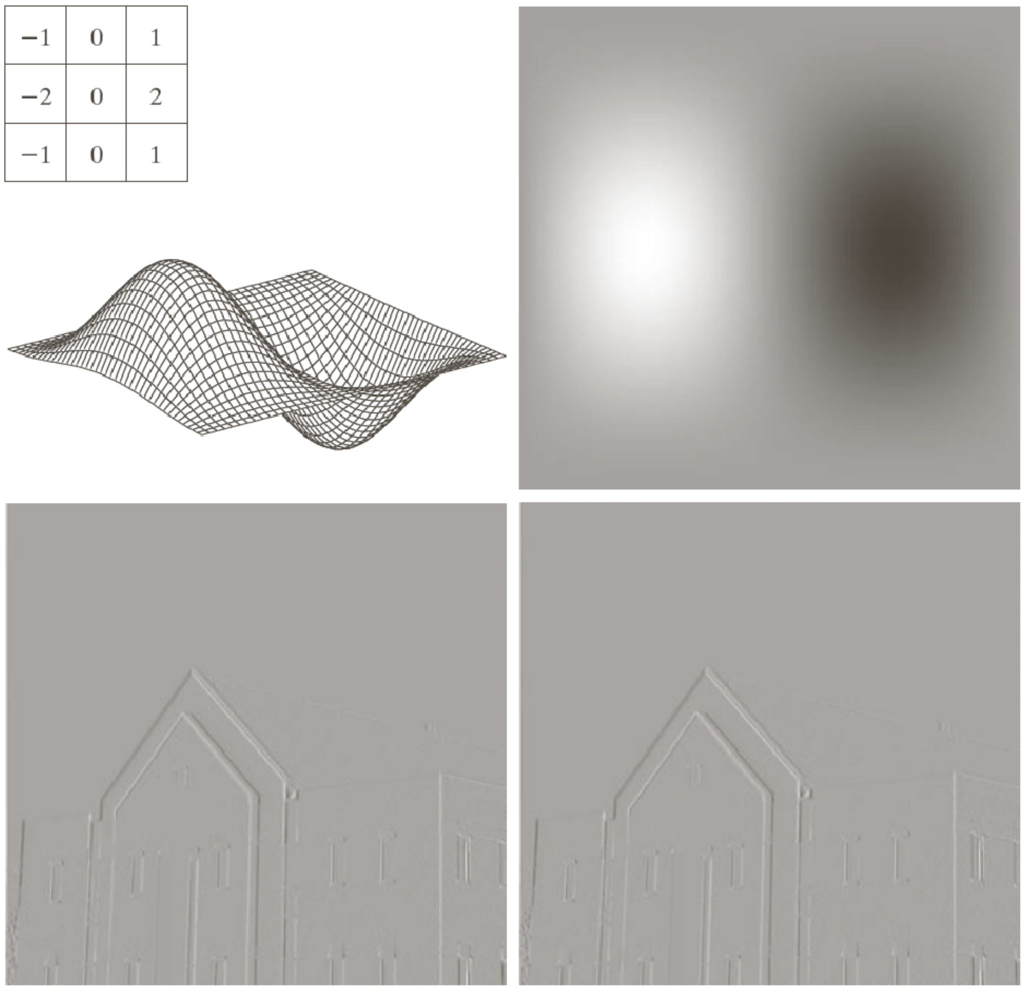
a b

FIGURE 4.38 (a) Image of a building, and (b) its spectrum.

Example 4.15

Obtaining a frequency domain filter from a small spatial mask

(Self study)



a b
c d

FIGURE 4.39 (a) A spatial mask and perspective plot of its corresponding frequency domain filter. (b) Filter shown as an image. (c) Result of filtering Fig. 4.38(a) in the frequency domain with the filter in (b). (d) Result of filtering the same image with the spatial filter in (a). The results are identical.

4.8 Image smoothing using freq domain filters

4.8.1 Ideal lowpass filters

$$H(u,v) = \begin{cases} 1, & \textbf{if } D(u,v) \leq D_0 \\ 0, & \textbf{if } D(u,v) > D_0 \end{cases} \qquad D(u,v) = \left[\, (u - P/2)^2 + (v - Q/2)^2 \,\right]^{1/2}$$

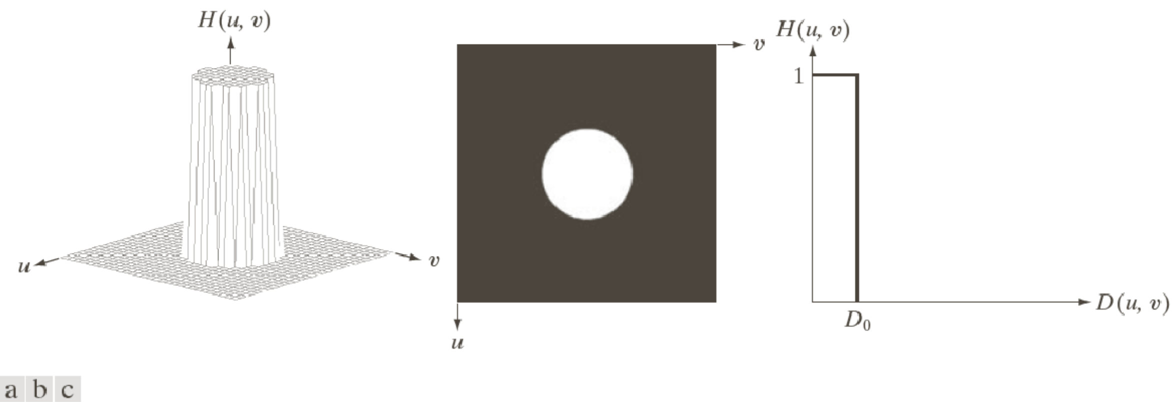


FIGURE 4.40 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

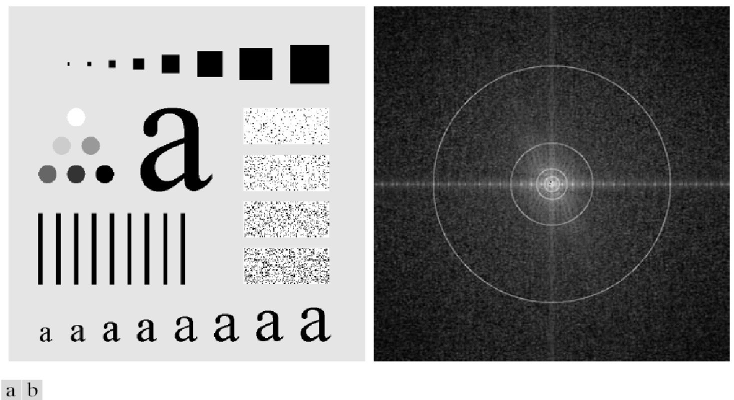
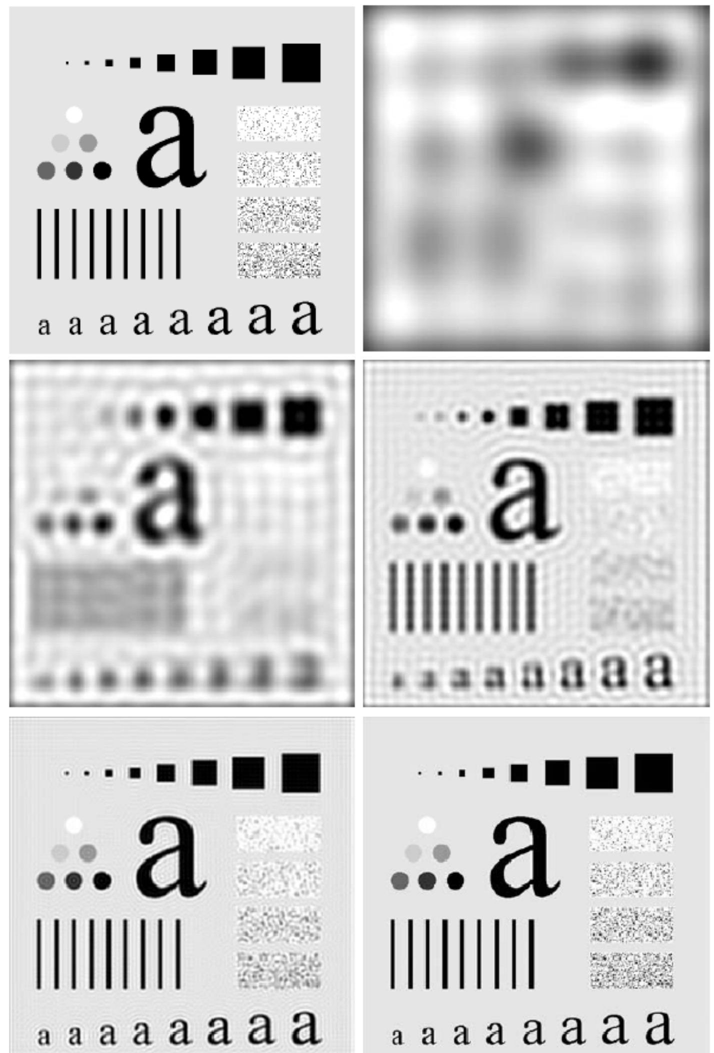


FIGURE 4.41 (a) Test pattern of size 688 $\times$ 688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.

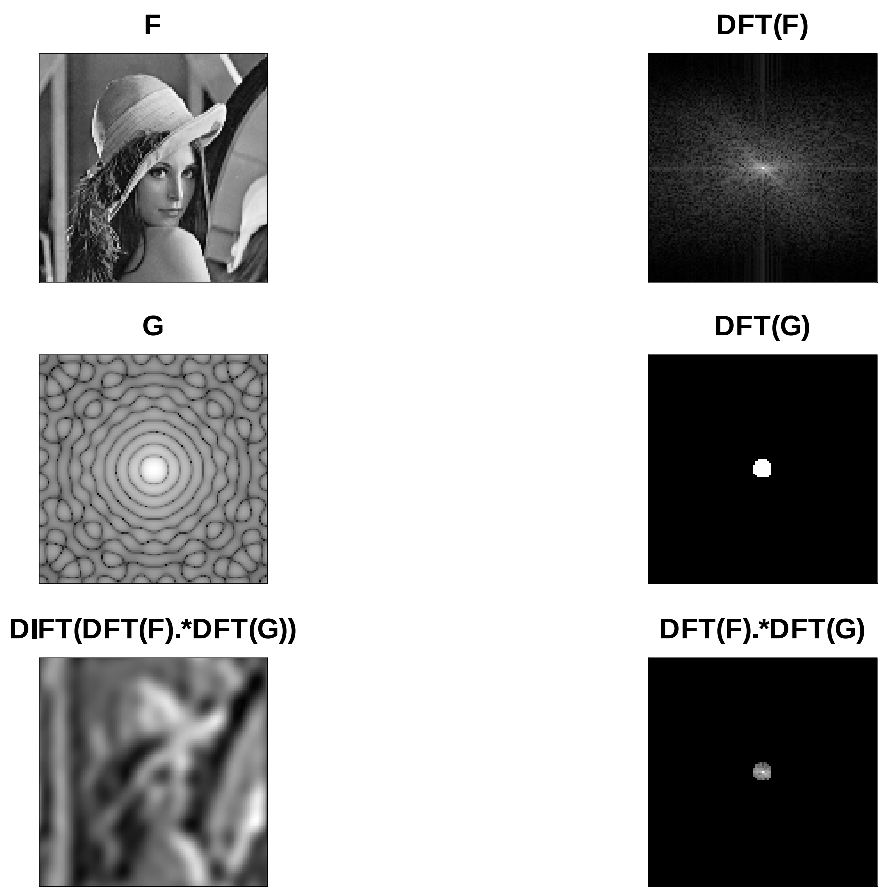
Ideal lowpass filter: The "ringing" phenomenon

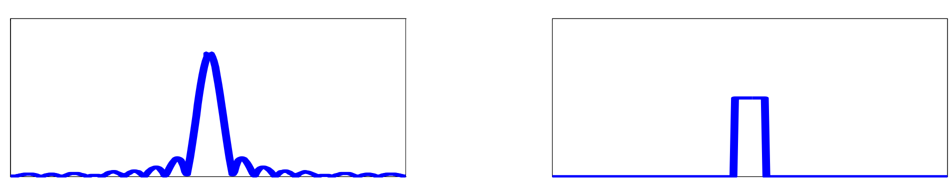


a b
c d
e f

FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

Explanation: Ideal lowpass filter $\Rightarrow$ Ringing





4.8.2 Butterworth lowpass filters

$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$$

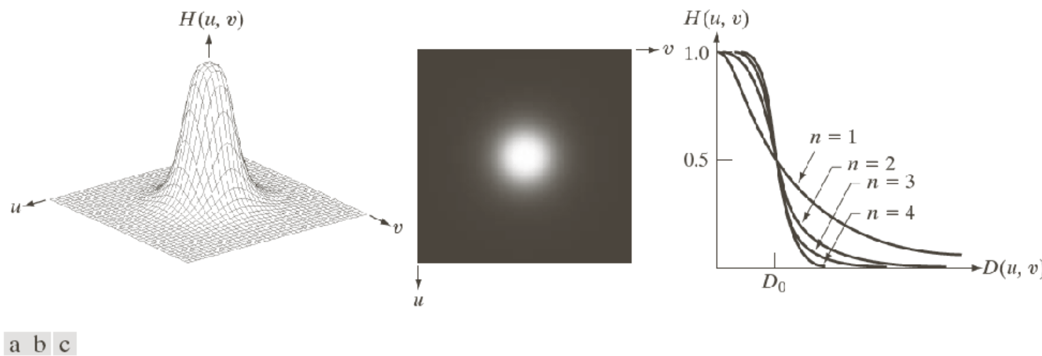


FIGURE 4.44 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

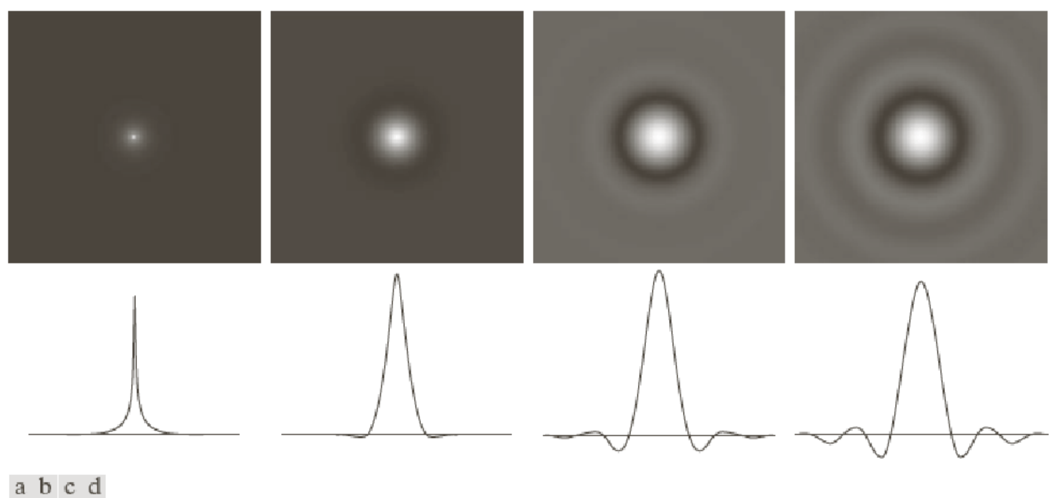
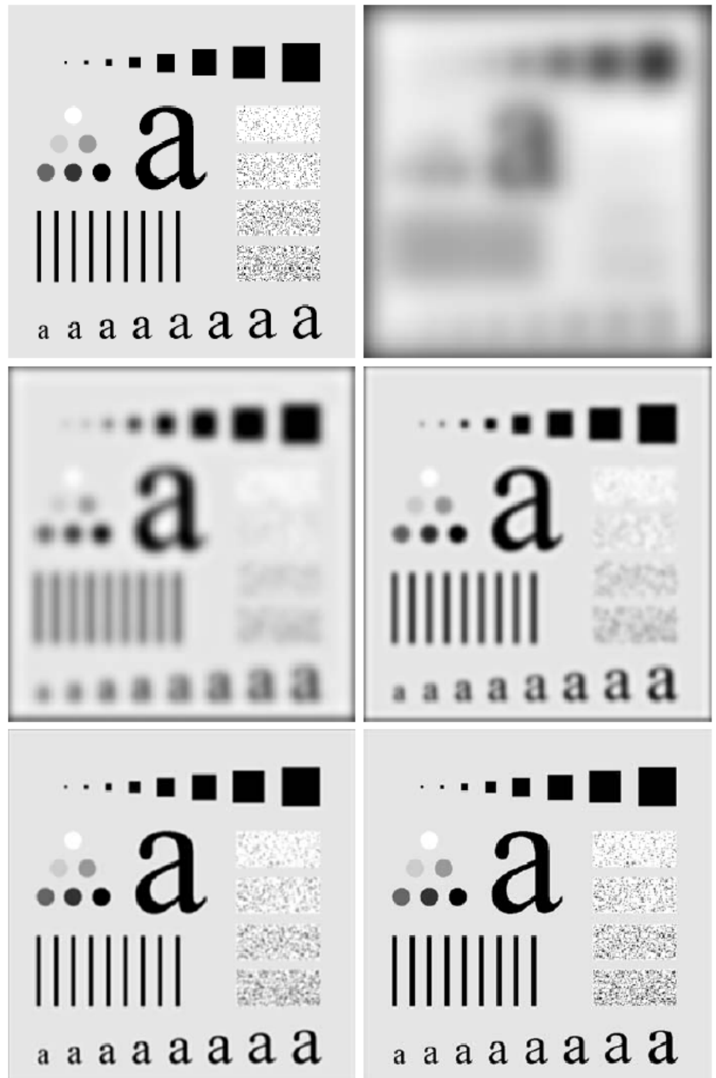


FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000 × 1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

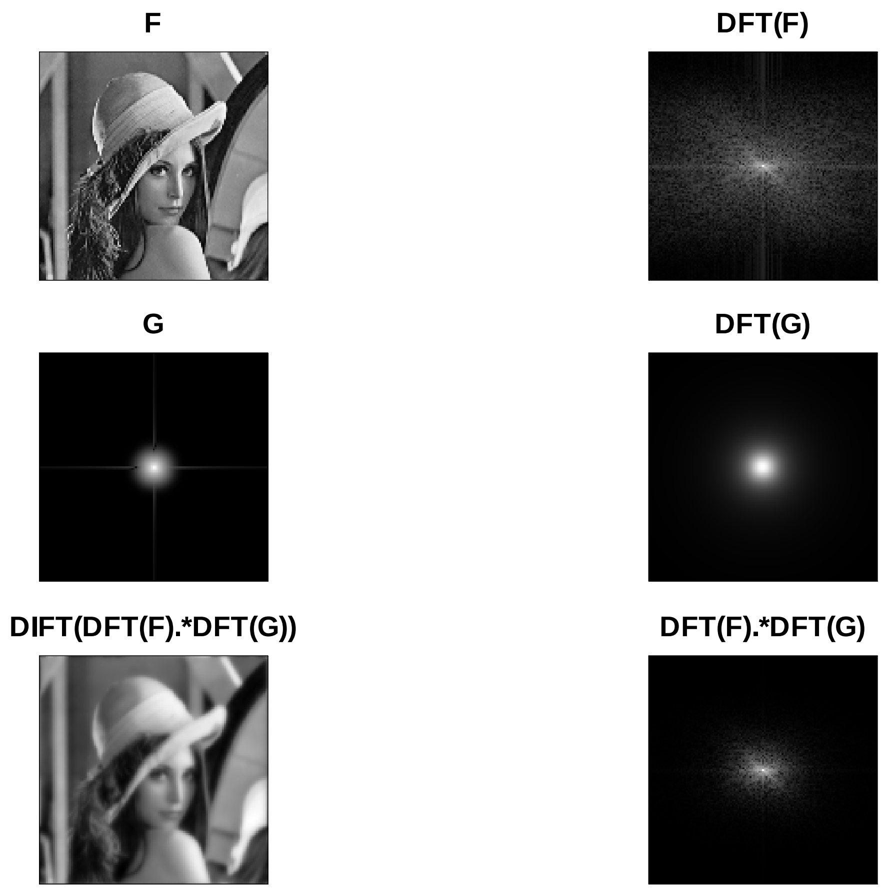
Butterworth lowpass filter with $n = 2$: Very little ringing

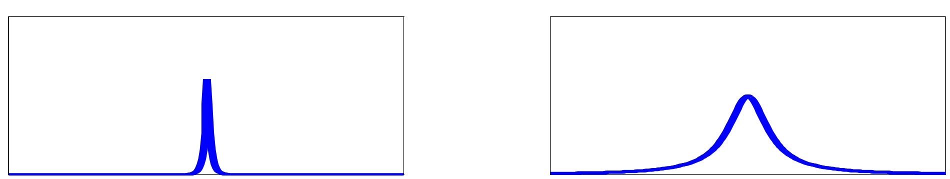


a b
c d
e f

FIGURE 4.45 (a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.

Explanation: Butterworth lowpass filter with $n = 1 \Rightarrow$ No ringing





4.8.3 Gaussian lowpass filters

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}, \qquad \textbf{when } \sigma = D_0$$

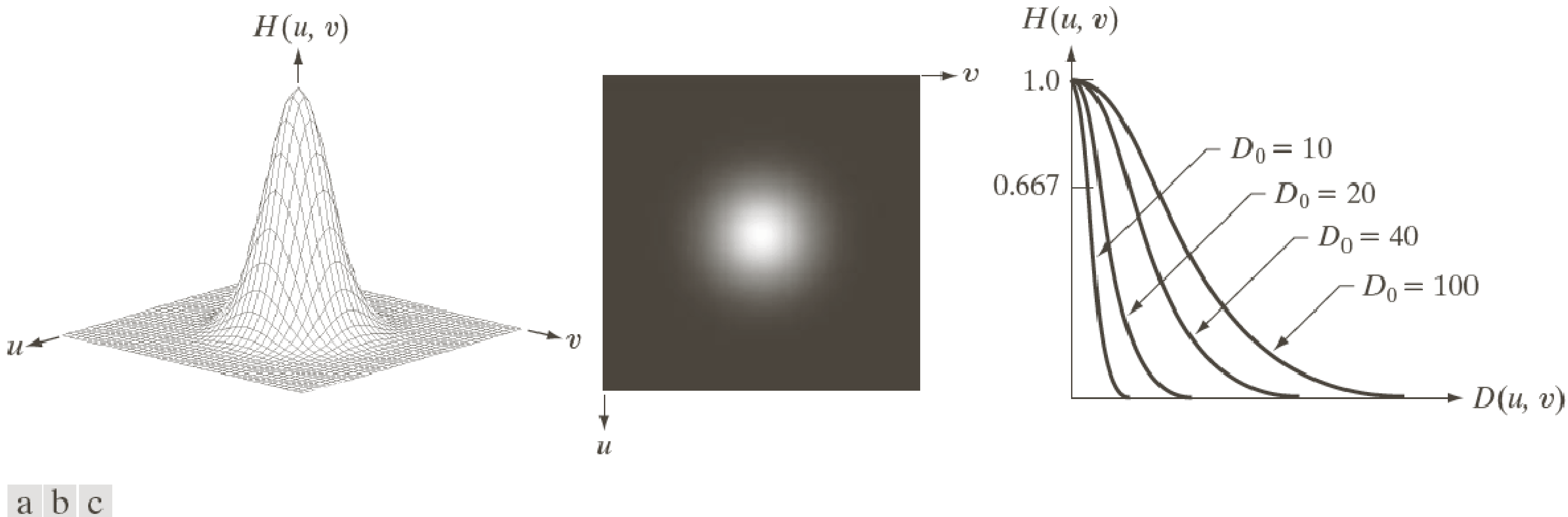
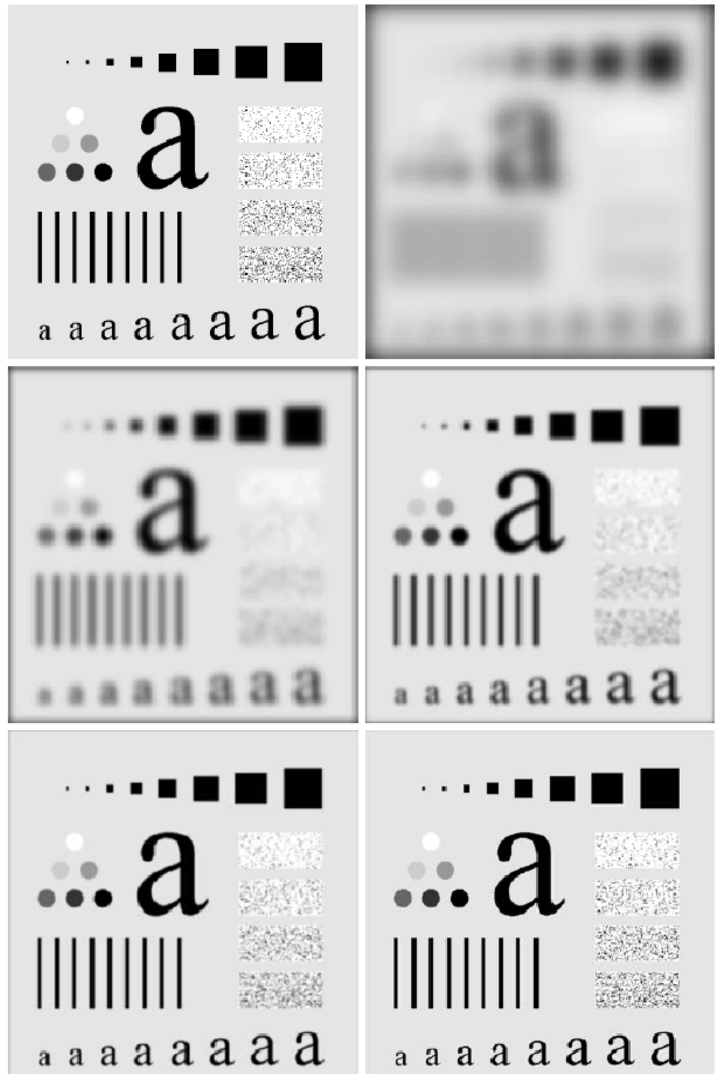


FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

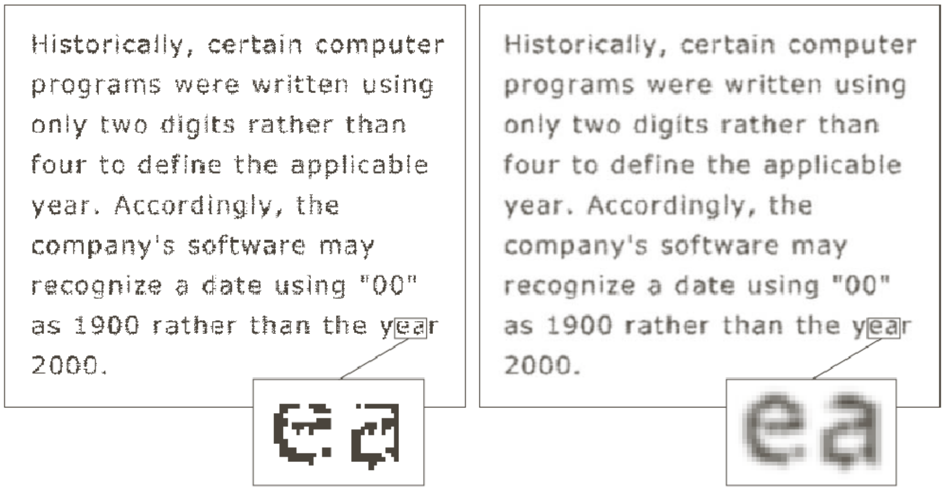
Gaussian lowpass filter $\Rightarrow$ No ringing



a b
c d
e f

FIGURE 4.48 (a) Original image. (b)–(f) Results of filtering using GLPFs with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Figs. 4.42 and 4.45.

4.8.4 Additional examples of lowpass filtering



a b

FIGURE 4.49 (a) Sample text of low resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

a b c

FIGURE 4.50 (a) Original image (784 × 732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).

4.9 Image sharpening using frequency domain filters

$$H_{\mathrm{HP}}(u,v) = 1 - H_{\mathrm{LP}}(u,v)$$

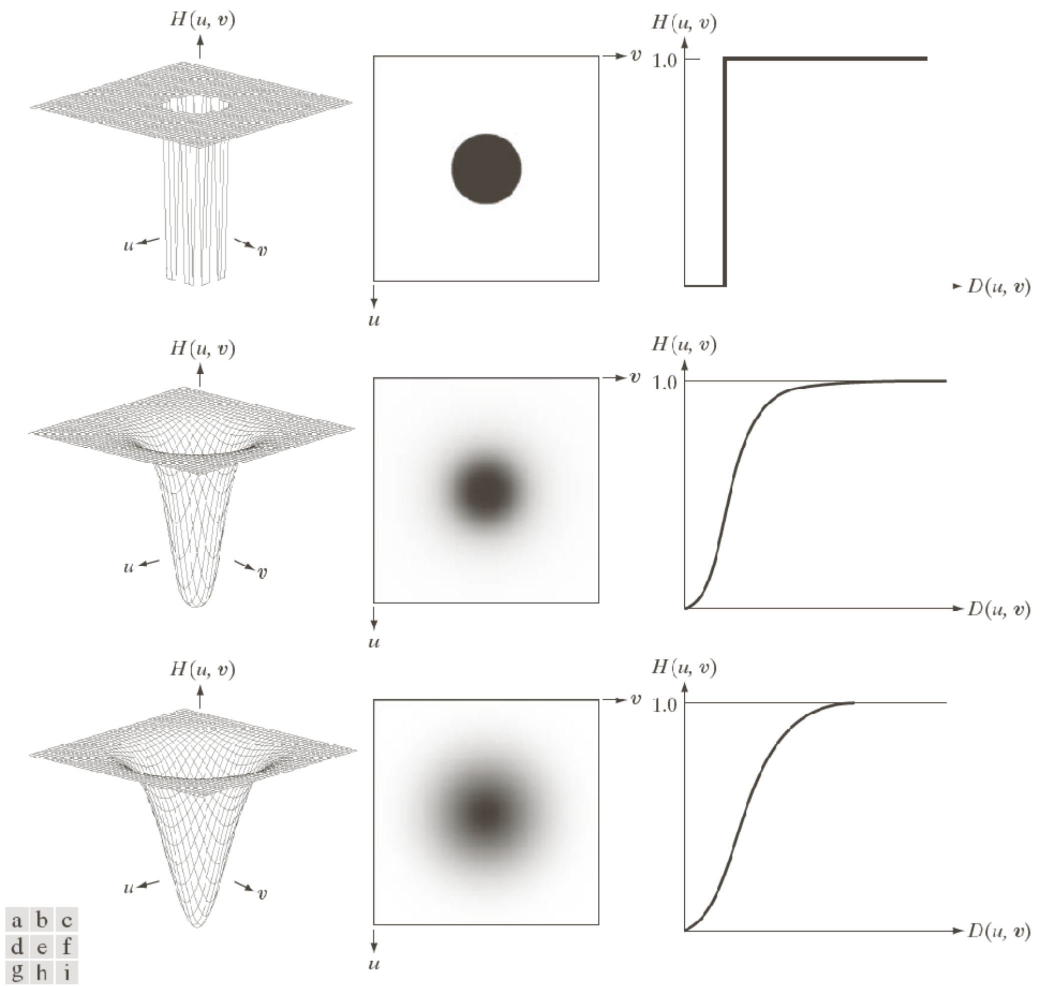
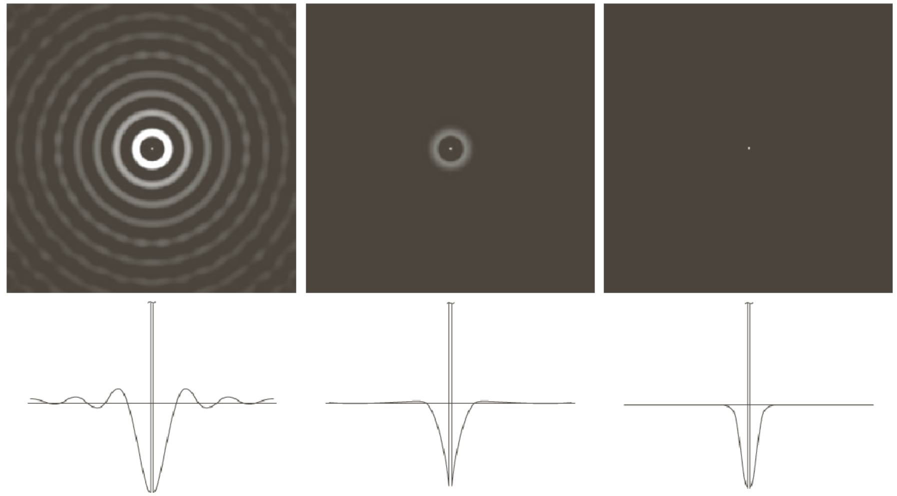


FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.



a b c

FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

4.9.1 Ideal highpass filters

$$H(u,v) = \begin{cases} 0, & \textbf{if } D(u,v) \leq D_0 \\ 1, & \textbf{if } D(u,v) > D_0 \end{cases}$$

a b c

FIGURE 4.54 Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30, 60$, and 160.

4.9.2 Butterworth highpass filters: $H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$

a b c

FIGURE 4.55 Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60$, and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

4.9.3 Gaussian highpass filters: $H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$

a b c

FIGURE 4.56 Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30, 60$, and 160, corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

Example 4.19: Highpass filtering & thresholding for image enhancement

a b c

FIGURE 4.57 (a) Thumb print. (b) Result of highpass filtering (a). (c) Result of thresholding (b). (Original image courtesy of the U.S. National Institute of Standards and Technology.)

4.9.4 The Laplacian in the frequency domain

$$\textbf{FT}\left\{\frac{d^n f(x)}{dx^n}\right\} = (iu)^n F(u)$$

$$\begin{aligned}\textbf{FT}\left\{\nabla^2 f(x,y)\right\} = \textbf{FT}\left\{\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}\right\} &= (iu)^2 F(u,v) + (iv)^2 F(u,v) \\ &= -(u^2+v^2)\, F(u,v)\end{aligned}$$

$$H(u,v) = -(u^2+v^2)$$

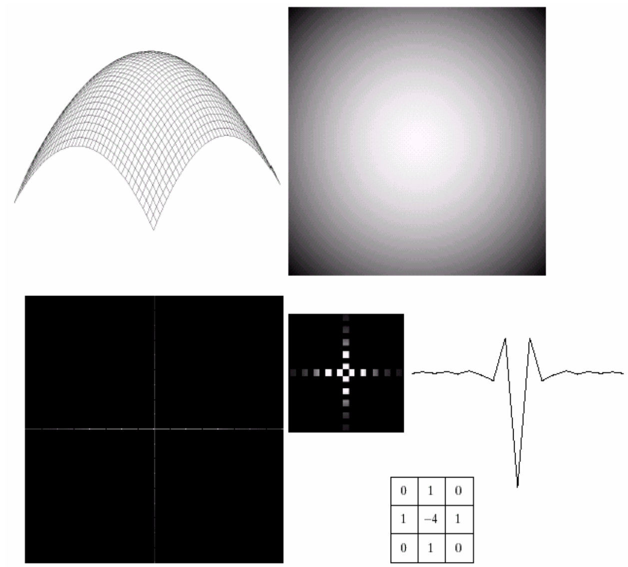
The center of the filter needs to be shifted, therefore

$$H(u,v) = -[\,(u-P/2)^2 + (v-Q/2)^2\,] = -D^2(u,v)$$

$$\nabla^2 f(x,y) \quad \Leftrightarrow \quad -D^2(u,v)\, F(u,v)$$

We form an enhanced image as follows, $g(x,y) = f(x,y) - \nabla^2 f(x,y)$**, which implies that**

$$g(x,y) = \textbf{IFT}\left\{\,[1 + D^2(u,v)]\; F(u,v)\,\right\}$$



a b
c d e
f

FIGURE 4.27 (a) 3-D plot of Laplacian in the frequency domain. (b) Image representation of (a). (c) Laplacian in the spatial domain obtained from the inverse DFT of (b). (d) Zoomed section of the origin of (c). (e) Gray-level profile through the center of (d). (f) Laplacian mask used in Section 3.7.

(Figure in Edition 2)

a b

FIGURE 4.58 (a) Original, blurry image. (b) Image enhanced using the Laplacian in the frequency domain. Compare with Fig. 3.38(e).

4.9.5 Unsharp masking, highboost filtering, high frequency emphasis filtering

$$g_{\text{mask}}(x,y) = f(x,y) - f_{\text{LP}}(x,y)$$

$$\begin{aligned} g(x,y) &= f(x,y) + k \cdot g_{\text{mask}}(x,y) \\ &= f(x,y) + k \cdot f(x,y) - k \cdot f_{\text{LP}}(x,y) \\ &= (k+1) \cdot f(x,y) - k \cdot f_{\text{LP}}(x,y) \end{aligned}$$

$$\begin{aligned} G(u,v) &= (k+1) \cdot F(u,v) - k \cdot H_{\text{LP}}(u,v) \cdot F(u,v) \\ &= (k+1-k \cdot H_{\text{LP}}(u,v)) \cdot F(u,v) \\ &= (k+1-k \cdot (1-H_{\text{HP}}(u,v))) \cdot F(u,v) \\ &= (1+k \cdot H_{\text{HP}}(u,v)) \cdot F(u,v) \end{aligned}$$

Unsharp Masking ($k = 1$)

$$H_{\text{US}}(u,v) = 1 + H_{\text{HP}}(u,v)$$

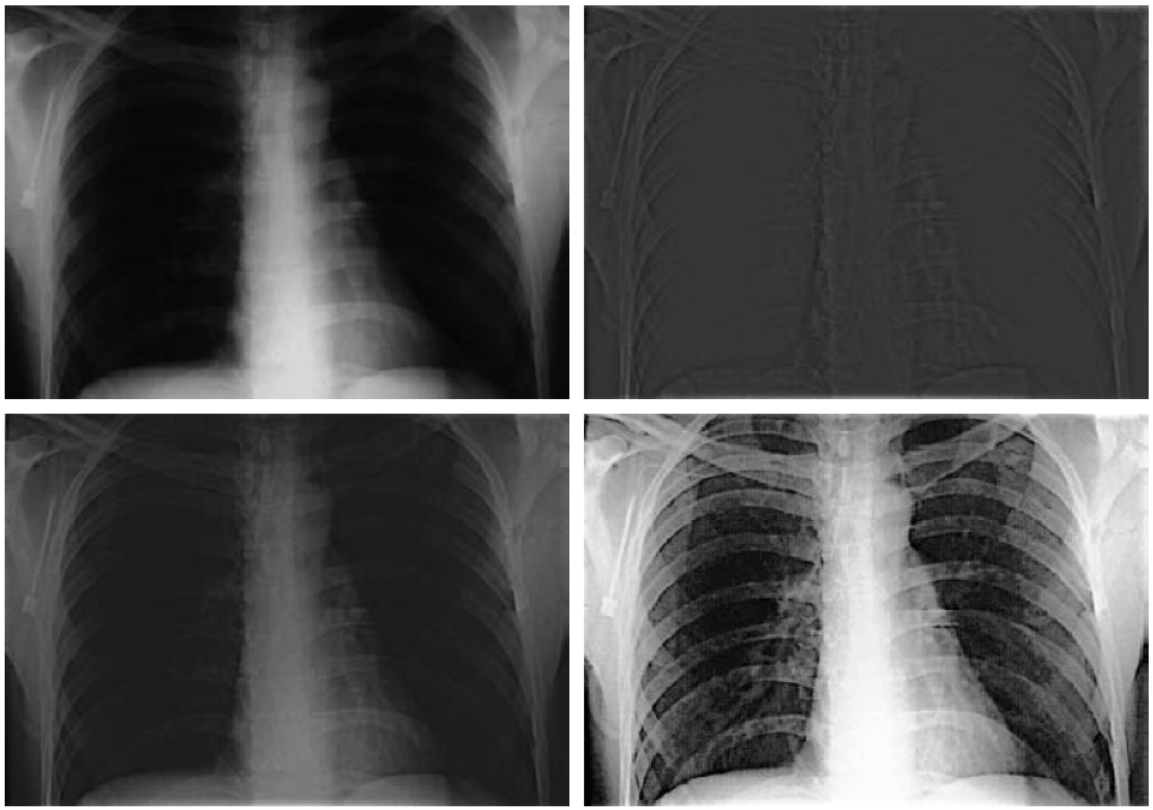
High-Boost Filtering ($k > 1$)

$$H_{\text{HB}}(u,v) = 1 + k \cdot H_{\text{HP}}(u,v)$$

High-Frequency Emphasis Filtering

$$H_{\text{HFE}}(u,v) = k_1 + k_2 \cdot H_{\text{HP}}(u,v), \quad k_1 \geq 0,\ k_2 \geq 0$$

Example 4.21: Image enhancement using high frequency emphasis filtering



a	b
c	d

FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

4.9.6 Homomorphic filtering

The Illumination-Reflectance Model (Section 2.3.4)

An image consists of two components:

(1) Illumination, that is how much light is shined unto the object,

$$0 \leq i(x, y) \leq \infty,$$

and is independent of the object itself;

(2) Reflectance, that is the fraction of light that is reflected by the object,

$$0 \leq r(x, y) \leq 1,$$

and is obviously a property of the object in question - when the object is black $r(x, y) = 0$ **- when the object is white** $r(x, y) = 1$

We therefore observe an image $f(x, y)$ **that is constructed as follows,**

$$f(x, y) = i(x, y)\, r(x, y),$$

after which the image is scaled so that $f(x, y) \in [0, L]$

We can not apply the FT to $i(x,y)$ and $r(x,y)$ separately, that is

$$\textbf{FT}\{\ f(x,y)\ \} \neq \textbf{FT}\{\ i(x,y)\ \} \times \textbf{FT}\{\ r(x,y)\ \},$$

so we improvise...

$$\begin{aligned} z(x,y) &= \ln\ f(x,y) \\ &= \ln[\ i(x,y)\ r(x,y)\ ] \\ &= \ln\ i(x,y) + \ln\ r(x,y) \end{aligned}$$

$$\begin{aligned} \textbf{FT}\{\ z(x,y)\ \} &= \textbf{FT}\{\ \ln\ i(x,y)\ \} + \textbf{FT}\{\ \ln\ r(x,y)\ \} \\ Z(u,v) &= F_i(u,v) + F_r(u,v) \end{aligned}$$

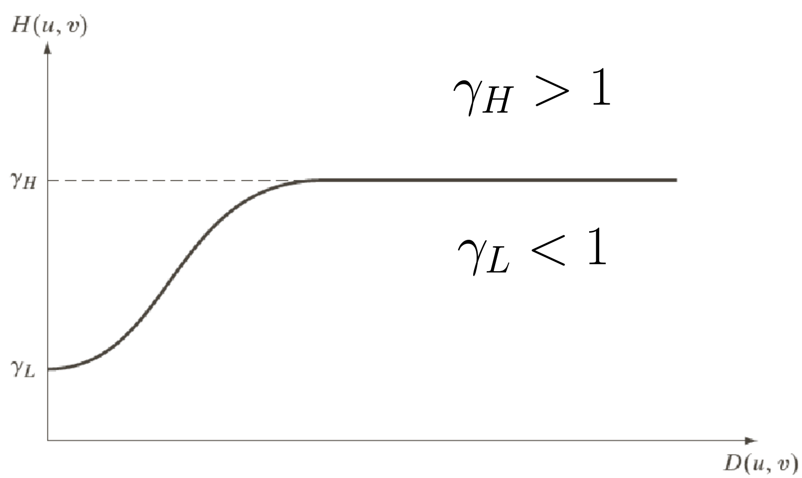


FIGURE 4.61 Radial cross section of a circularly symmetric homomorphic filter function. The vertical axis is at the center of the frequency rectangle and $D(u, v)$ is the distance from the center.

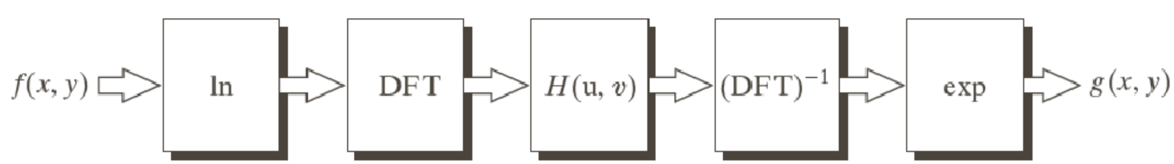
Application of the homomorphic filter, $H(u,v)$:

$$
\begin{aligned}
S(u,v) &= H(u,v)\ Z(u,v) \\
&= H(u,v)\ F_i(u,v) + H(u,v)\ F_r(u,v)
\end{aligned}
$$

$$
\begin{aligned}
s(x,y) &= \textbf{IFT}\{\ S(u,v)\ \} \\
&= \textbf{IFT}\{\ H(u,v)\ F_i(u,v)\ \} + \textbf{IFT}\{\ H(u,v)\ F_r(u,v)\ \} \\
&= i'(x,y) + r'(x,y)
\end{aligned}
$$

$$
\begin{aligned}
g(x,y) &= e^{s(x,y)} \\
&= e^{i'(x,y)+r'(x,y)} \\
&= e^{i'(x,y)}\ e^{r'(x,y)} \\
&= i_0(x,y)\ r_0(x,y)
\end{aligned}
$$

FIGURE 4.60 Summary of steps in homomorphic filtering.



<u>ILLUMINATION:</u> Associated with low freq coef's, that is slow varying spatial intensities

<u>REFLECTANCE:</u> Associated with high freq coef's, that is fast varying spatial intensities

Therefore the effect of the homomorphic filter is as follows:

(1) **it attenuates the low frequency coefficients and therefore makes the illumination more uniform, which removes artifacts like shadows**

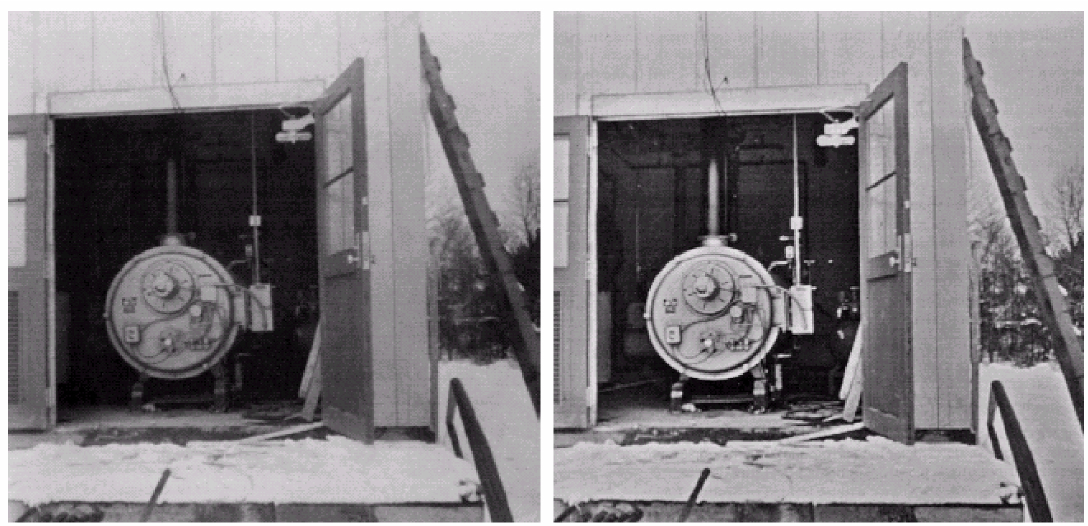
(2) **it emphasizes the high frequency coefficients and therefore makes the image sharper**

A homomorphic filter can be constructed as follows,

$$H(u,v) = (\gamma_H - \gamma_L)\left(1 - e^{-c(D^2(u,v)/D_0^2)}\right) + \gamma_L$$

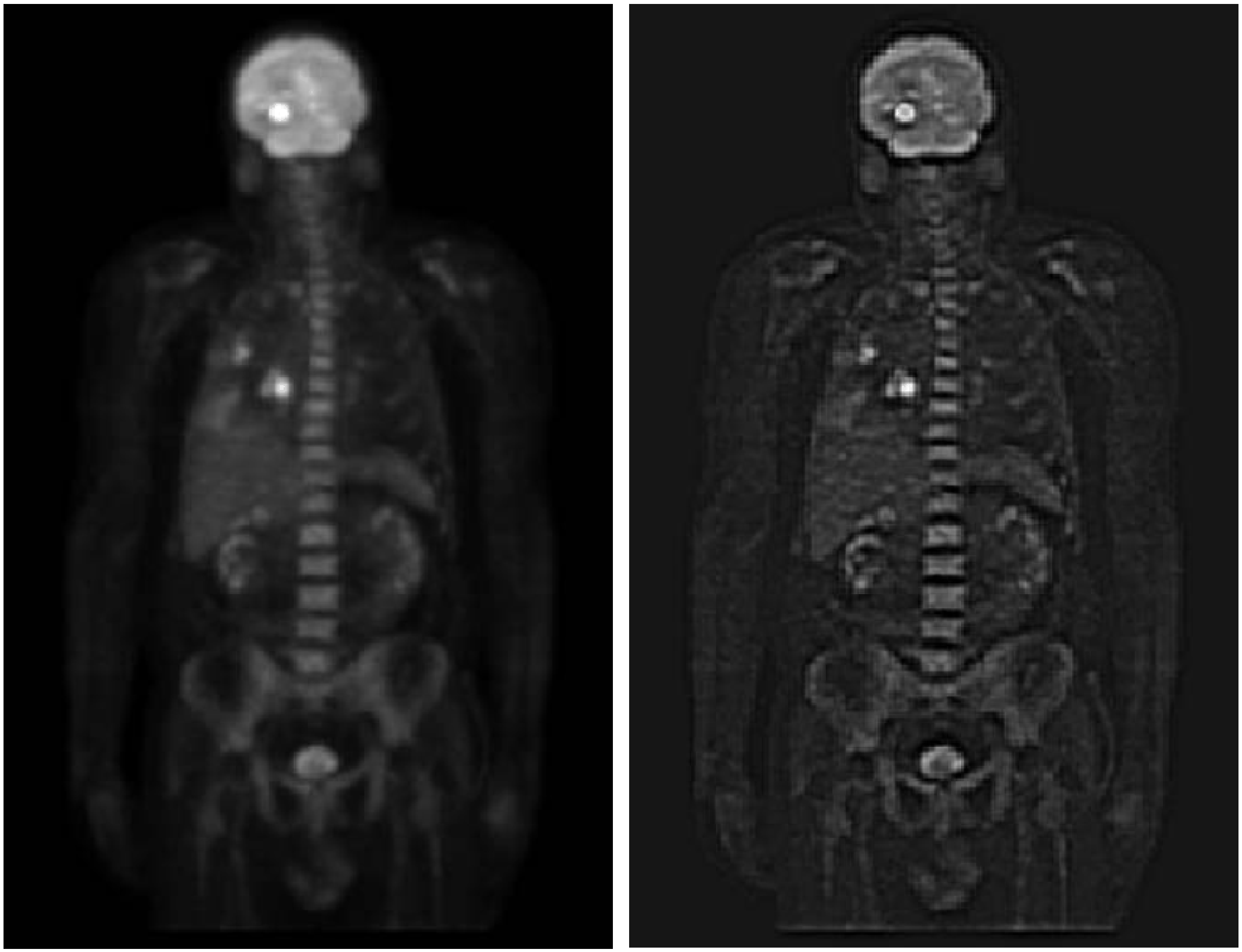
a b

FIGURE 4.33 (a) Original image. (b) Image processed by homomorphic filtering (note details inside shelter). (Stockham.)



(Fig in Edition 2)

Example 4.22: Image enhancement using homomorphic filtering



a b

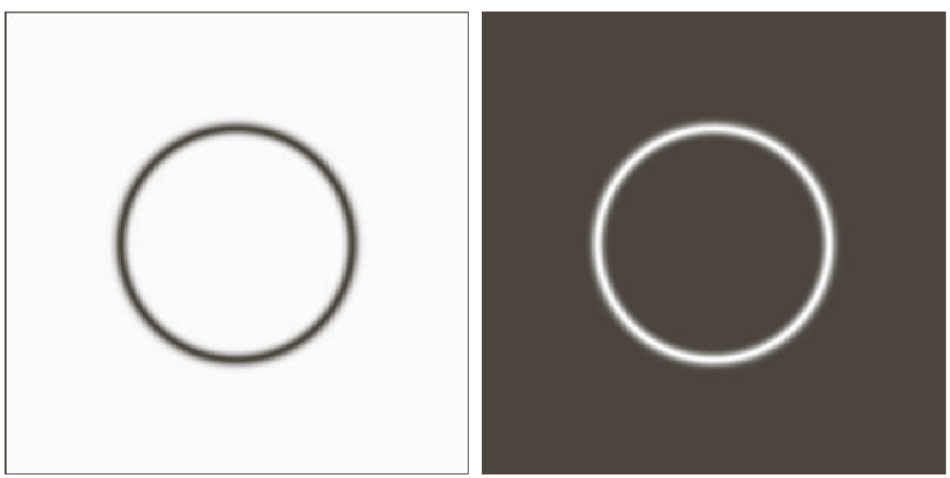
FIGURE 4.62 (a) Full body PET scan. (b) Image enhanced using homomorphic filtering. (Original image courtesy of Dr. Michael E. Casey, CTI PET Systems.)

4.10 Selective filtering

4.10.1 Bandreject and bandpass filters: $H_{\mathrm{BP}}(u,v) = 1 - H_{\mathrm{BR}}(u,v)$

TABLE 4.6
Bandreject filters. W is the width of the band, D is the distance $D(u,v)$ from the center of the filter, D_0 is the cutoff frequency, and n is the order of the Butterworth filter. We show D instead of $D(u,v)$ to simplify the notation in the table.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \le D \le D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u,v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$	$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$



a b
FIGURE 4.63
(a) Bandreject Gaussian filter. (b) Corresponding bandpass filter. The thin black border in (a) was added for clarity; it is not part of the data.

4.10.2 Notch filters

Notch reject filter: $H_{\mathrm{NR}}(u,v) = \prod_{k=1}^{Q} H_k(u,v)H_{-k}(u,v)$

$H_k(u,v)$: **Highpass filter centered at** (u_k, v_k)

$H_{-k}(u,v)$: **Highpass filter centered at** $(-u_k, -v_k)$

Distance computations:

$$D_k(u,v) = \left[(u - M/2 - u_k)^2 + (v - N/2 - v_k)^2\right]^{1/2}$$

$$D_{-k}(u,v) = \left[(u - M/2 + u_k)^2 + (v - N/2 + v_k)^2\right]^{1/2}$$

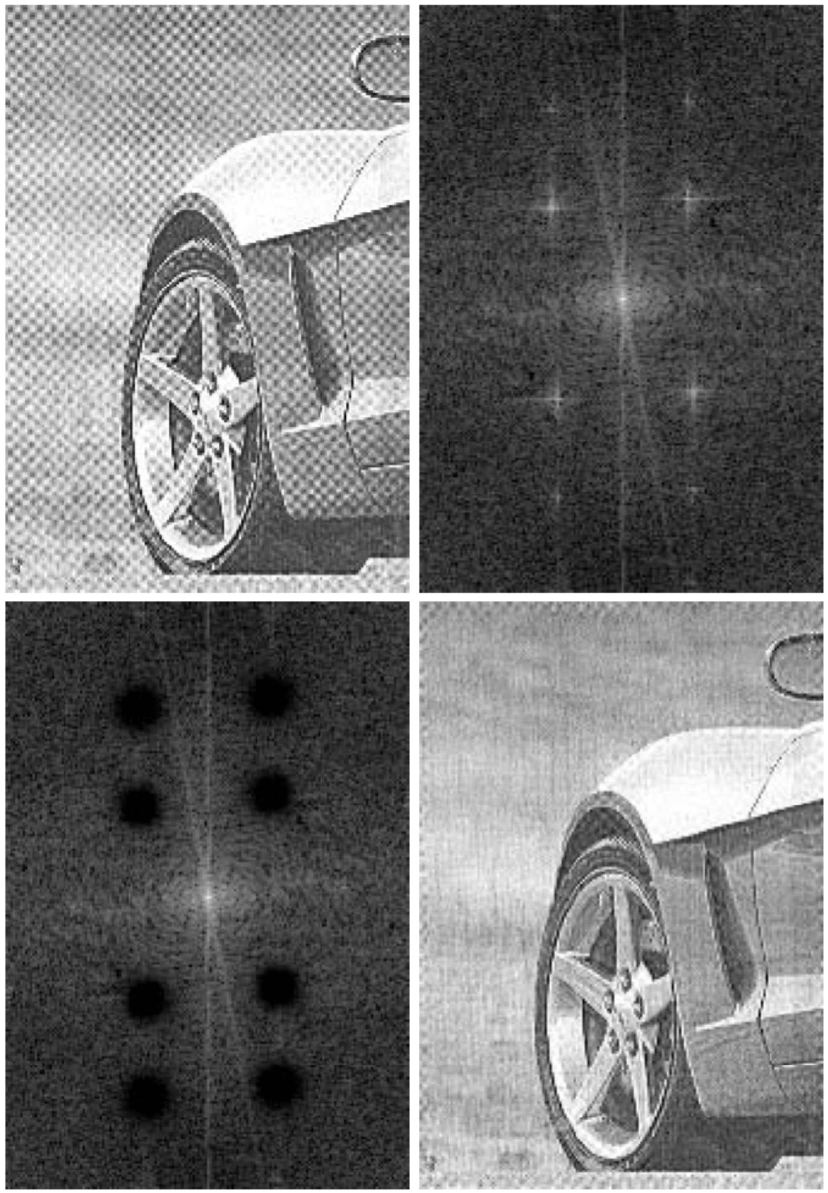
Butterworth notch reject filter of order n**, containing** $Q = 3$ **notch pairs:**

$$H_{\mathrm{NR}}(u,v) = \prod_{k=1}^{3} \left\{\frac{1}{1 + [D_{0k}/D_k(u,v)]^{2n}}\right\} \left\{\frac{1}{1 + [D_{0k}/D_{-k}(u,v)]^{2n}}\right\}$$

Notch pass filter: $H_{\mathrm{NP}}(u,v) = 1 - H_{\mathrm{NR}}(u,v)$

Note: Benefit of not padding outweigh cost of wraparound error

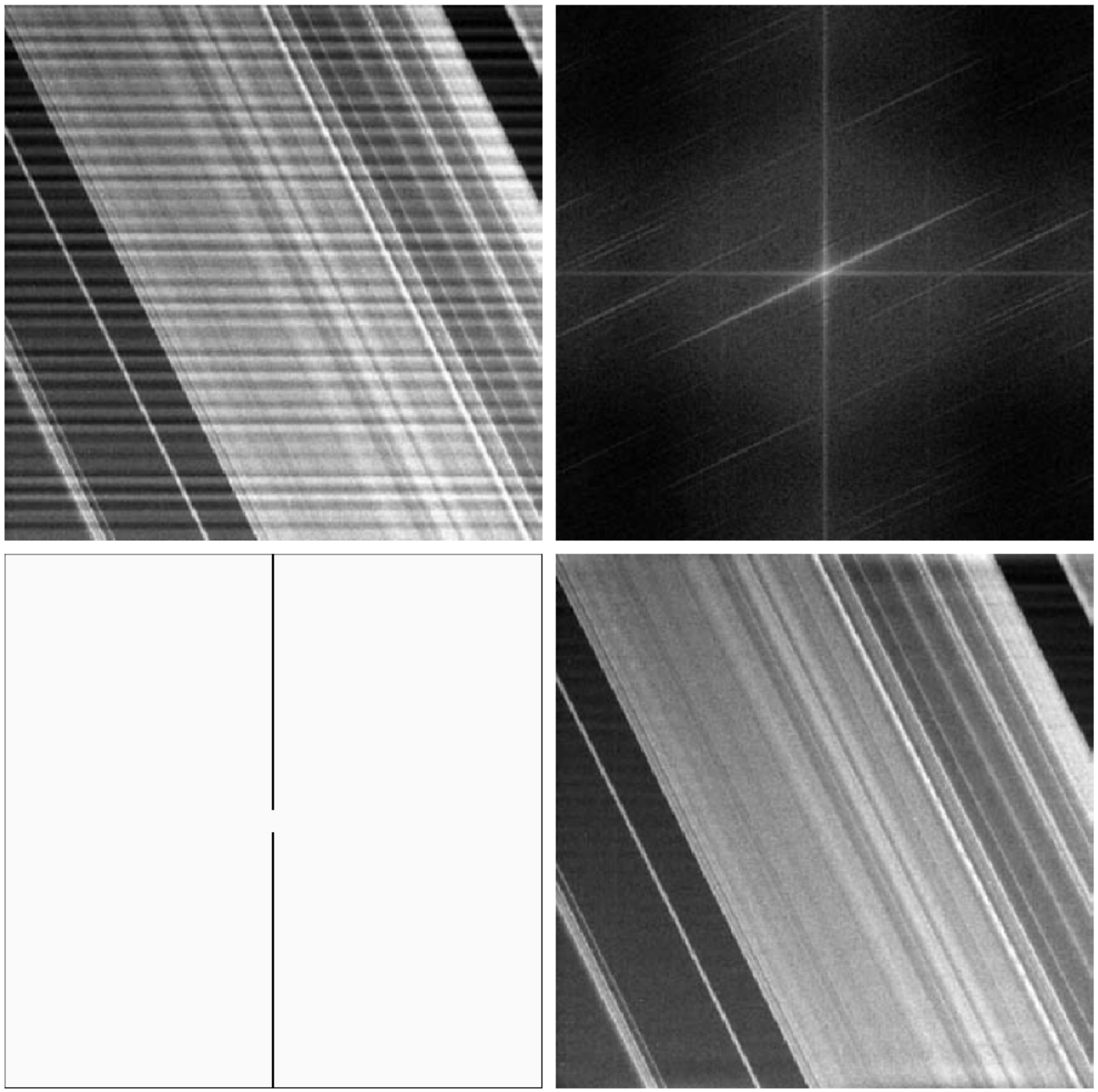
Example 4.23: Reduction of moiré patterns using notch filtering



a	b
c	d

FIGURE 4.64 (a) Sampled newspaper image showing a moiré pattern. (b) Spectrum. (c) Butterworth notch reject filter multiplied by the Fourier transform. (d) Filtered image.

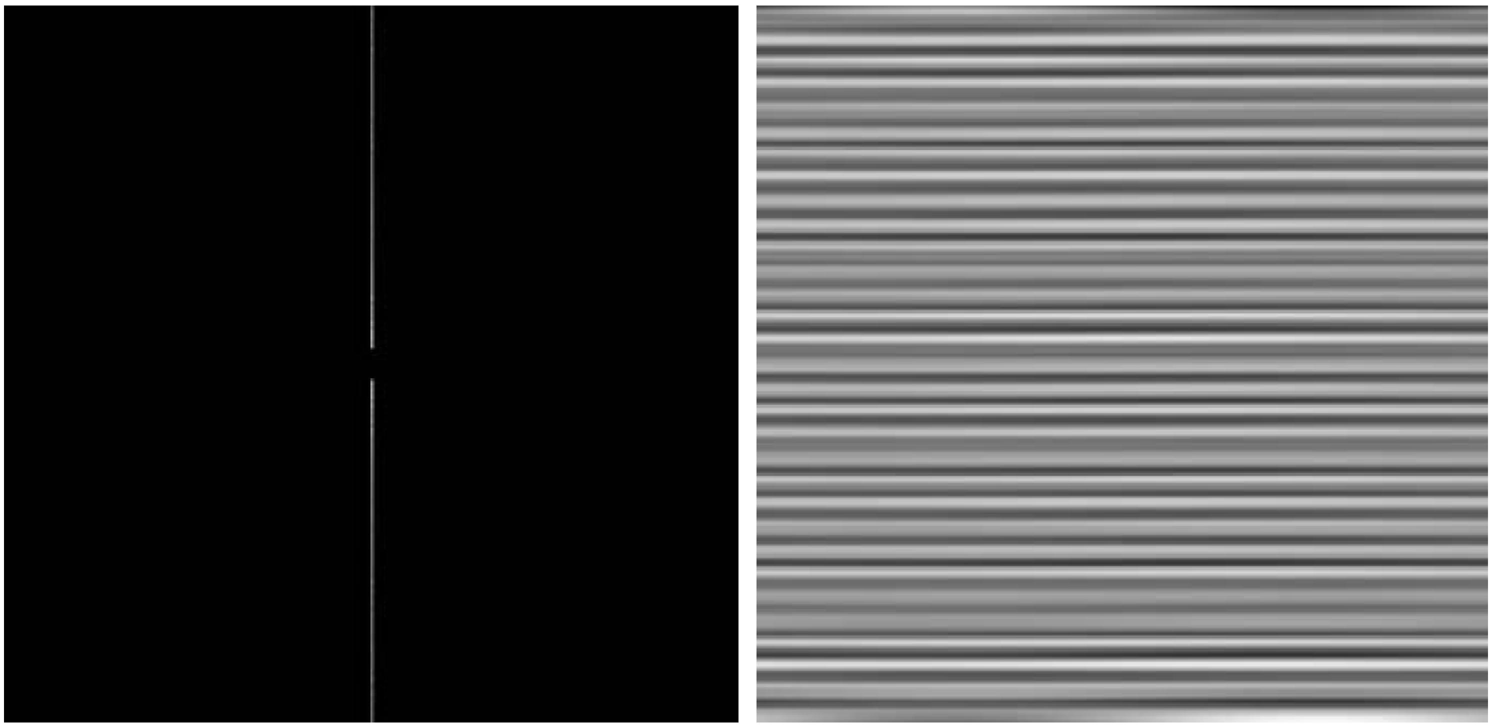
Example 4.24: Enhancement of corrupted Cassini Saturn image using notch filtering



a b
c d

FIGURE 4.65 (a) 674×674 image of the Saturn rings showing nearly periodic interference. (b) Spectrum: The bursts of energy in the vertical axis near the origin correspond to the interference pattern. (c) A vertical notch reject filter. (d) Result of filtering. The thin black border in (c) was added for clarity; it is not part of the data. (Original image courtesy of Dr. Robert A. West, NASA/JPL.)

Example 4.24: Enhancement of corrupted Cassini Saturn image using notch filtering... Explanation



a b

FIGURE 4.66 (a) Result (spectrum) of applying a notch pass filter to the DFT of Fig. 4.65(a). (b) Spatial pattern obtained by computing the IDFT of (a).