CONVOLUTION

Continuous convolution in one dimension

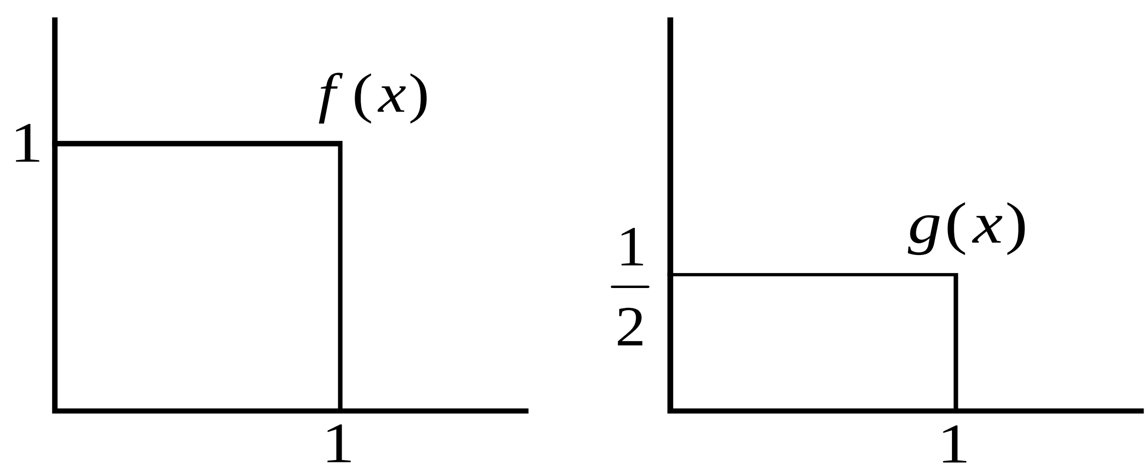
Definition: The convolution of $f(x)$ and $g(x)$ is

$$(f \bigstar g)(x) = \int_{-\infty}^{\infty} f(\alpha) g(x - \alpha)\, d\alpha,$$

where α is the integration variable

Example 6: Calculate the convolution of

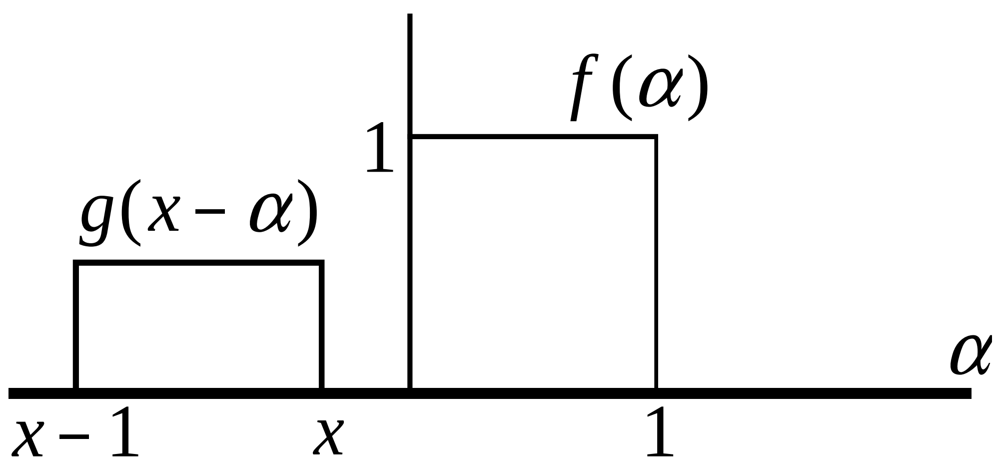
$$f(x) = \begin{cases} 1, & \textbf{if} \quad x \in [0,1] \\ 0, & \textbf{otherwise} \end{cases} \quad \textbf{and} \quad g(x) = \begin{cases} \frac{1}{2}, & \textbf{if} \quad x \in [0,1] \\ 0, & \textbf{otherwise} \end{cases}$$



$$\begin{aligned} g(x-\alpha) &= \begin{cases} \frac{1}{2}, & \textbf{if} \quad x-\alpha \in [0,1] \\ 0, & \textbf{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{2}, & \textbf{if} \quad \alpha \in [x-1,x] \\ 0, & \textbf{otherwise} \end{cases} \end{aligned}$$

Note that for different values of x, $f(\alpha)$ remains constant while $g(x-\alpha)$ varies

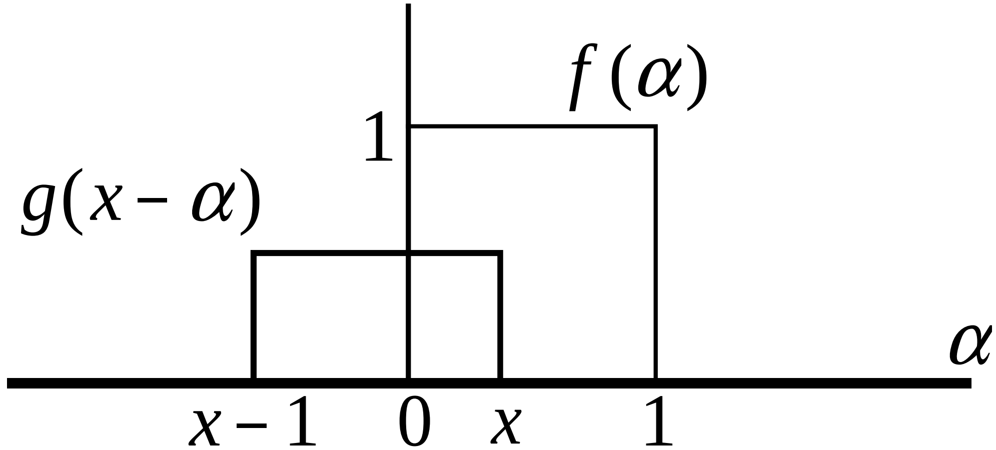
- **If $x \in (-\infty, 0]$,**



then

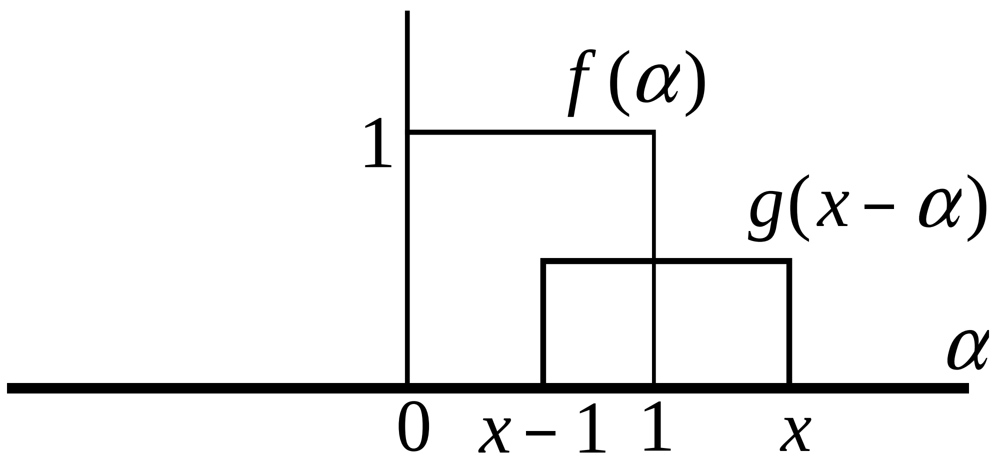
$$(f\bigstar g)(x) = \int_{-\infty}^{\infty} f(\alpha)g(x-\alpha)\,d\alpha = 0$$

- **If** $x \in [0, 1]$,



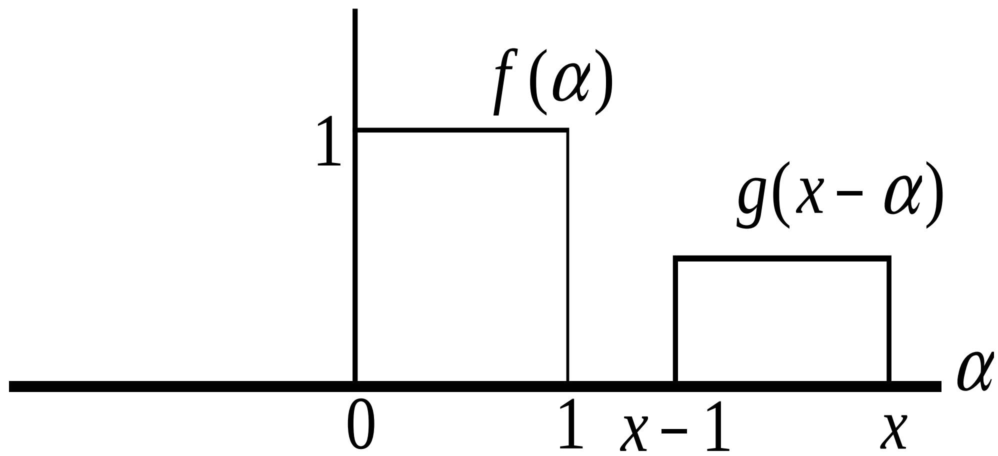
$$(f \bigstar g)(x) = \int_0^x (\frac{1}{2})(1)\, d\alpha = \frac{x}{2}$$

- **If** $x \in [1, 2]$,



$$(f \bigstar g)(x) = \int_{x-1}^1 \frac{1}{2}(1)\, d\alpha = 1 - \frac{x}{2}$$

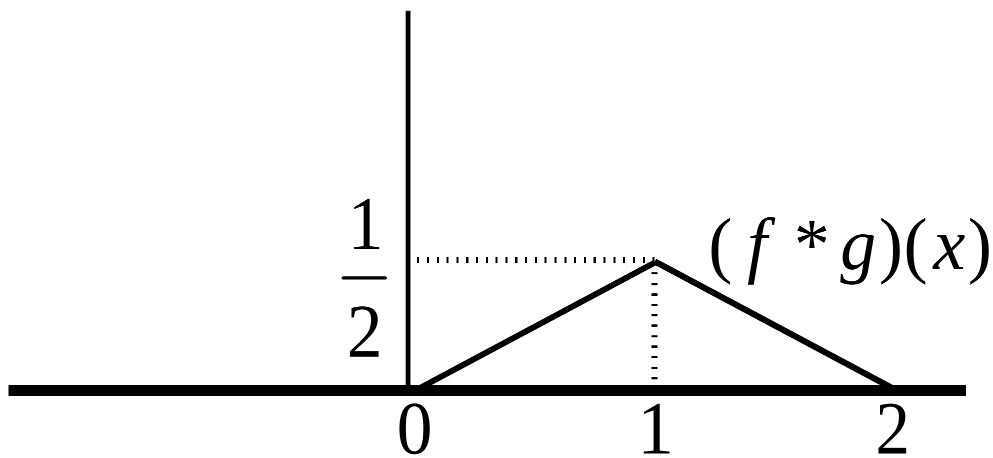
- **If** $x \in [2, \infty)$**,**



$$(f \bigstar g)(x) = 0$$

Therefore

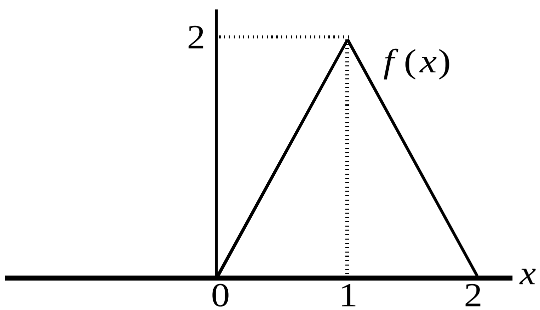
$$(f \bigstar g)(x) = \begin{cases} \frac{x}{2}, & \textbf{if} \quad x \in [0,1] \\ 1 - \frac{x}{2}, & \textbf{if} \quad x \in [1,2] \\ 0, & \textbf{otherwise} \end{cases}$$

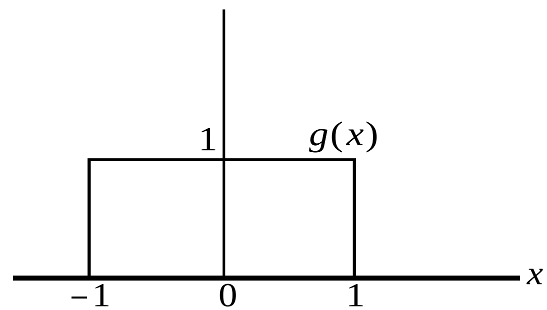


Smoother?

Example 7: **Calculate the convolution of**

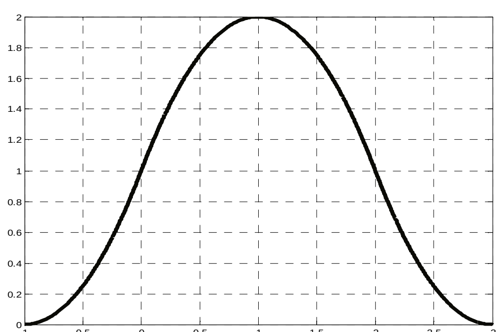
$$f(x) = \begin{cases} 2x, & \textbf{if} \quad x \in [0,1] \\ 2(2-x), & \textbf{if} \quad x \in [1,2] \\ 0, & \textbf{otherwise} \end{cases} \quad \textbf{and} \quad g(x) = \begin{cases} 1, & \textbf{if} \quad x \in [-1,1] \\ 0, & \textbf{otherwise} \end{cases}$$





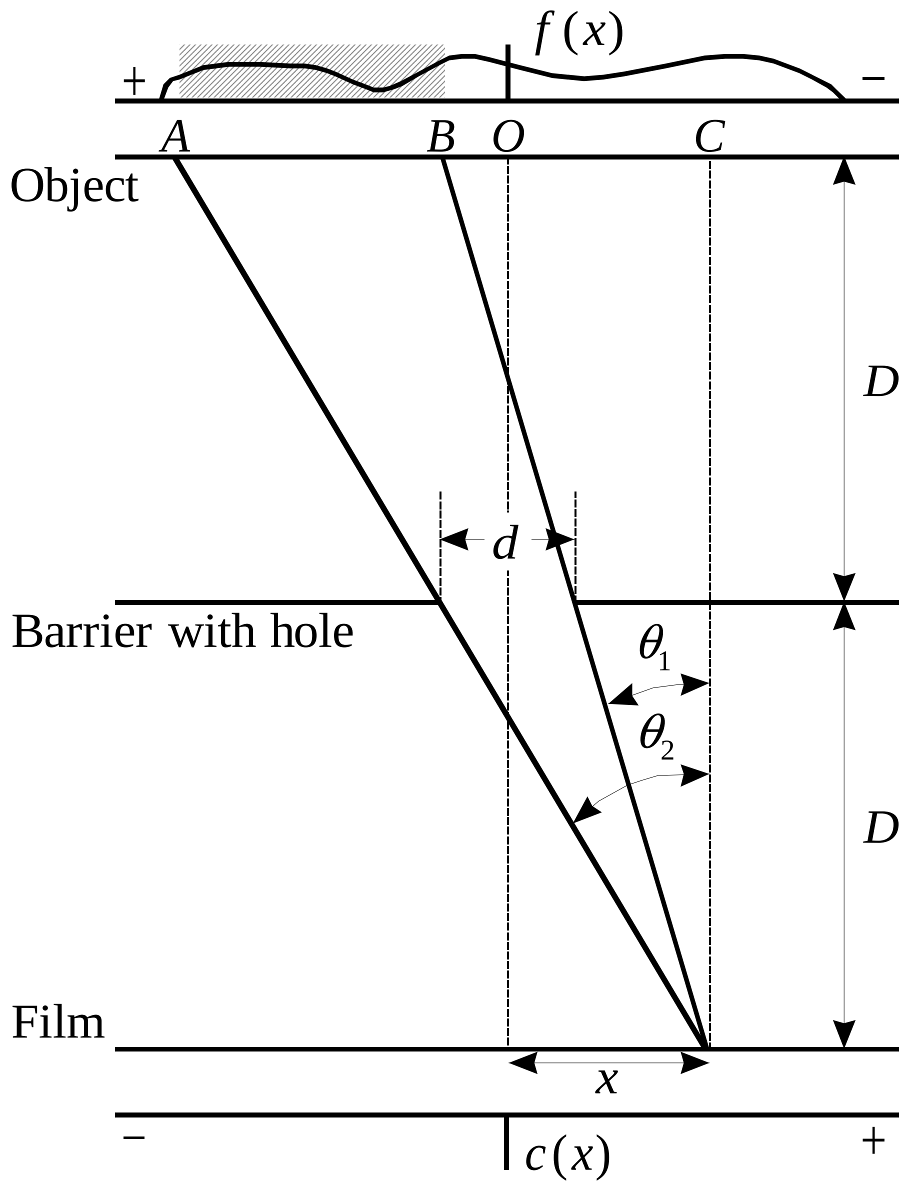
Solution: (Verify at home)

$$(f\bigstar g)(x) = \begin{cases} 0, & \textbf{if} \quad x \in (-\infty,-1] \\ (x+1)^2, & \textbf{if} \quad x \in [-1,0] \\ -x^2+2x+1, & \textbf{if} \quad x \in [0,2] \\ (x-3)^2, & \textbf{if} \quad x \in [2,3] \\ 0, & \textbf{if} \quad x \in [3,\infty) \end{cases}$$



Again smoothing!

Application of convolution: The simple camera



$f(x) \equiv$ **light intensity of the object at** x

$c(x) \equiv$ **light intensity of the photo (on film) at** x

Since we initially assume that we have no lens, the camera is out of focus and $c(x) \neq f(x)$

$$\tan\theta_1 = \frac{x - d/2}{D} = \frac{BC}{2D} \quad \Rightarrow \quad BC = 2x - d$$

$$\tan\theta_2 = \frac{x + d/2}{D} = \frac{AC}{2D} \quad \Rightarrow \quad AC = 2x + d$$

$$OA = AC - x = (2x + d) - x = x + d$$

$$OB = BC - x = (2x - d) - x = x - d$$

$$\begin{aligned} c(x) &= \textbf{cumulation of } f(x) \textbf{ between } x - d \textbf{ and } x + d \\ &= \int_{x-d}^{x+d} f(x)\,dx \\ &= \int_{x-d}^{x+d} f(s)\,ds \end{aligned}$$

Before we define the transfer function, $g(x)$, we first introduce the Dirac Delta function, $\delta(x)$:

$$\delta(x) = \begin{cases} \infty, & \textbf{if } x = 0 \\ 0, & \textbf{if } x \neq 0 \end{cases}$$

This function has the following three important properties:

$$\textbf{(1)} \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

$$\textbf{(2)} \int_{-\infty}^{\infty} f(x)\, \delta(x - x_0)\, dx = f(x_0)$$

$$\textbf{(3)} \ (f \bigstar \delta)(x) = f(x)$$

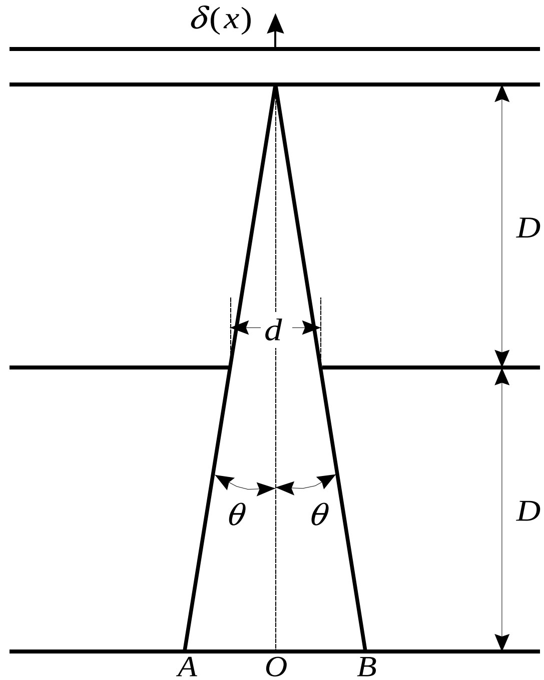
Also note that when $g(x) = \delta(x+T) + \delta(x) + \delta(x-T)$,

$$\begin{aligned} (f \bigstar g)(x) &= \int_{-\infty}^{\infty} f(\alpha)\, g(x-\alpha)\, d\alpha \\ &= \int_{-\infty}^{\infty} f(\alpha)\, \delta[(x+T)-\alpha]\, d\alpha + \int_{-\infty}^{\infty} f(\alpha)\, \delta[x-\alpha]\, d\alpha \\ &\quad + \int_{-\infty}^{\infty} f(\alpha)\, \delta[(x-T)-\alpha]\, d\alpha \\ &= f(x+T) + f(x) + f(x-T) \end{aligned}$$

Therefore, in order to obtain $g(x)$, we have to take a photo of a light impulse, that is "a speck of light"

This implies that $f(x) = \delta(x)$ and that

$$(f \bigstar g)(x) = (\delta \bigstar g)(x) = (g \bigstar \delta)(x) = g(x)$$



$$\tan\theta = \frac{OA}{2D} = \frac{d/2}{D} \quad \Rightarrow \quad OA = d \qquad\qquad \tan\theta = \frac{OB}{2D} = \frac{d/2}{D} \quad \Rightarrow \quad OB = d$$

Therefore

$$g(x) = \begin{cases} 1, & x \in [-d, d] \\ 0, & \textbf{otherwise} \end{cases}$$

and

$$\begin{aligned} (f \bigstar g)(x) &= \int_{-\infty}^{\infty} f(\alpha)\, g(x-\alpha)\, d\alpha \\ &= \int_{x-d}^{x+d} f(\alpha)\, d\alpha \\ &= \int_{x-d}^{x+d} f(s)\, ds \\ &= c(x) \end{aligned}$$

which implies that $c(x) = (f \bigstar g)(x)$

Therefore, for the photo to be perfect, we have to ensure that $g(x) = \delta(x)$**.**

Note that, the wider $g(x)$ **is, the more 'blurred' the photo** $c(x)$ **will become**

Now: Can we reconstruct $f(x)$ **from** $c(x)$ **and** $g(x)$**?... In theory we can, using the convolution theorem, that is**

$$\textbf{FT}\left\{(f \bigstar g)(x)\right\} = F(u)\, G(u)$$

Proof of convolution theorem

$$\begin{aligned}\textbf{FT}\,\{(f\bigstar g)(x)\} &= \int_{-\infty}^{\infty} \{(f\bigstar g)(x)\}\; e^{-2\pi i u x}\, dx \\ &= \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} f(\alpha)\; g(x-\alpha) d\alpha\right\}\; e^{-2\pi i u x}\, dx\end{aligned}$$

Let $x-\alpha=t$, **then** $x=t+\alpha$ **and** $dx=dt$. **Therefore**

$$\begin{aligned}\textbf{FT}\,\{(f\bigstar g)(x)\} &= \int_{-\infty}^{\infty} f(\alpha)\; \left\{\int_{-\infty}^{\infty} g(t)\; e^{-2\pi i u(\alpha+t)} dt\right\}\; d\alpha \\ &= \int_{-\infty}^{\infty} f(\alpha)\; \left\{\int_{-\infty}^{\infty} g(t)\; e^{-2\pi i u t}\, dt\right\}\; e^{-2\pi i u \alpha} d\alpha \\ &= \int_{-\infty}^{\infty} f(\alpha)\; G(u)\; e^{-2\pi i u \alpha}\, d\alpha \\ &= G(u) \int_{-\infty}^{\infty} f(\alpha)\; e^{-2\pi i u \alpha}\, d\alpha \\ &= F(u)\; G(u) \\ &\longrightarrow\end{aligned}$$

Prove the following at home: $(f\bigstar g)(x) = (g\bigstar f)(x)$; **FT** $\{f \times g\} = (F\bigstar G)(u)$

Since

$$c(x) = (f \bigstar g)(x),$$

the convolution theorem implies that

$$C(u) = F(u)\, G(u)$$

Therefore

$$f(x) = \textbf{IFT}\left\{\frac{C(u)}{G(u)}\right\} \quad \equiv \quad \textbf{the recovered original image}$$

This process is called "deconvolution" or "deblurring". A method similar to the discrete two-dimensional equivalent of this method was used to partially recover photos that were taken by the "Hubble" telescope during the period that the telescope had a faulty lens.

Why is it not possible to fully recover the original function $f(x)$?

Note that we can also explain the blurring of $f(x)$ in Fourier space:

$$C(u) = F(u)\,G(u)$$

The function $G(u)$ is a so-called “lowpass filter” which retains the coefficients of the low frequency modes in $F(u)$, but attenuates the coefficients of the high frequency modes. This results in the smoothing (in this particular case, blurring) of $f(x)$ in the physical space

Discrete convolution in one dimension

Definition: Consider two vectors, $f(x)$ and $g(x)$, where

$$\begin{aligned} f(x) &= [\,f(0), \quad f(1), \quad f(2), \quad \ldots, \quad f(N-1)\,] \\ g(x) &= [\,g(0), \quad g(1), \quad g(2), \quad \ldots, \quad g(N-1)\,] \end{aligned}$$

with $f(x+N) = f(x) \quad \forall\, x$, and $g(x+N) = g(x) \quad \forall\, x$, then

$$(f \bigstar g)(x) = \frac{1}{N} \sum_{n=0}^{N-1} f(n)\, g(x-n)$$

We assume that $f(x)$ and $g(x)$ have the same dimension, N

Note that $(f\bigstar g)(x+N) = (f\bigstar g)(x) \quad \forall\, x$ (verify)

Prove the three convolution theorems for the discrete case!

Smoothing convolution masks

Consider a mask

$$g = [\, g(-1), \quad g(0), \quad g(1)\,]\,,$$

where $\sum g = 1$

(this condition guarantees a lowpass filter in Fourier space ... later)

Therefore

$$c(x) = g(-1)f(x-1) + g(0)f(x) + g(1)f(x+1)$$

Example 8

If $g = [\,\frac{1}{4}, \quad \frac{1}{2}, \quad \frac{1}{4}\,]$, then $c(x) = \frac{1}{4}(f(x-1) + 2f(x) + f(x+1))$

Let $c(x) = (G\bigstar f)(x)$ **and then find** G **(**N **dimensional)**

$$\begin{aligned}
c(x) &= (G\bigstar f)(x) \\
&= \frac{1}{N}\sum_{n=0}^{N-1} G(n)f(x-n) \\
&= \frac{1}{N}\{G(0)f(x) + G(1)f(x-1) + G(2)f(x-2)+ \\
&\quad \ldots + G(N-1)f(x-N+1)\} \\
&= \frac{1}{N}\{G(0)f(x) + G(1)f(x-1) + G(2)f(x-2)+ \\
&\quad \ldots + G(N-1)f(x+1)\} \\
&= g(-1)f(x-1) + g(0)f(x) + g(1)f(x+1)
\end{aligned}$$

This implies that

$$\begin{aligned}
G(0) &= Ng(0) \\
G(1) &= Ng(-1) \\
G(2) &= G(3) = \ldots = G(N-2) = 0 \\
G(N-1) &= Ng(1)
\end{aligned}$$

Therefore

$$G = N\,[\,g(0), \quad g(-1), \quad 0, \quad \ldots, \quad 0, \quad g(1)\,]$$

Let FT $\{G(x)\} = \hat{G}(u)$**, then**

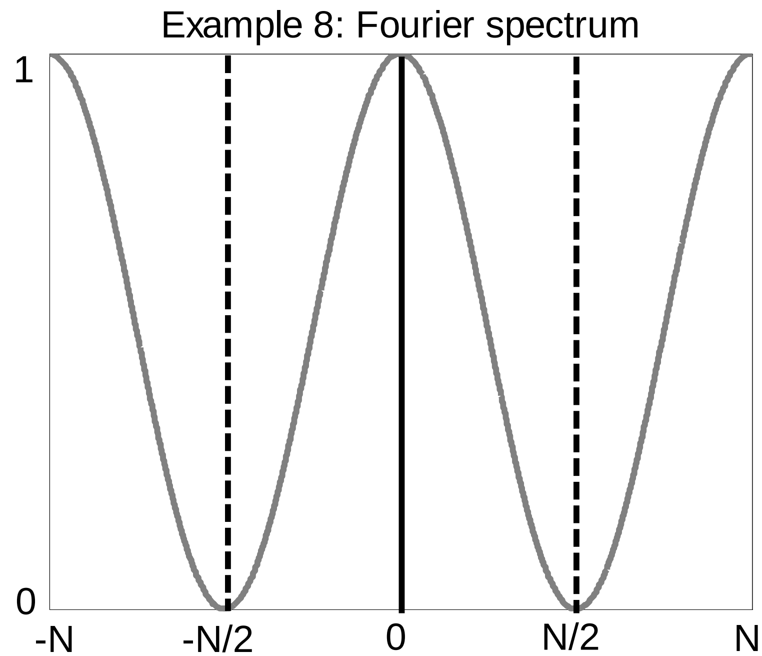
$$\begin{aligned}
\hat{G}(u) &= \frac{1}{N}\sum_{x=0}^{N-1} G(x)e^{-2\pi iux/N} \\
&= \frac{1}{N}\left\{G(0) + G(1)e^{-2\pi iu/N} + 0 + \ldots + 0 + G(N-1)e^{-2\pi iu(N-1)/N}\right\} \\
&= \frac{1}{N}\left\{Ng(0) + Ng(-1)e^{-2\pi iu/N} + 0 + \ldots + 0 + Ng(1)e^{2\pi iu/N}\right\} \\
&= g(0) + g(-1)e^{-2\pi iu/N} + g(1)e^{2\pi iu/N}
\end{aligned}$$

Example 8 (continued...)

$$\begin{aligned}
\hat{G}(u) &= \frac{1}{4}\left\{2 + e^{2\pi iu/N} + e^{-2\pi iu/N}\right\} \\
&= \frac{1}{4}\{2 + 2\cos(2\pi u/N)\} \\
&= \frac{1}{2}\{1 + \cos(2\pi u/N)\} \\
&= \cos^2(\pi u/N)
\end{aligned}$$

Note that $\hat{G}(0) = 1$ **... this is the reason for the condition,** $\sum g = 1$

Also note that $\hat{G}(N/2) = \hat{G}(-N/2) = 0$



This is a LOWPASS FILTER!

Example 9

When $g = [\,\frac{1}{3}, \quad \frac{1}{3}, \quad \frac{1}{3}\,]$**, we can show that**

$$\hat{G}(0) = 1 \quad \textbf{and} \quad \left|\,\hat{G}(N/2)\,\right| = \left|\,\hat{G}(-N/2)\,\right| = \frac{1}{3},$$

which also implies a LOWPASS FILTER **(verify)**

Sharpening convolution masks

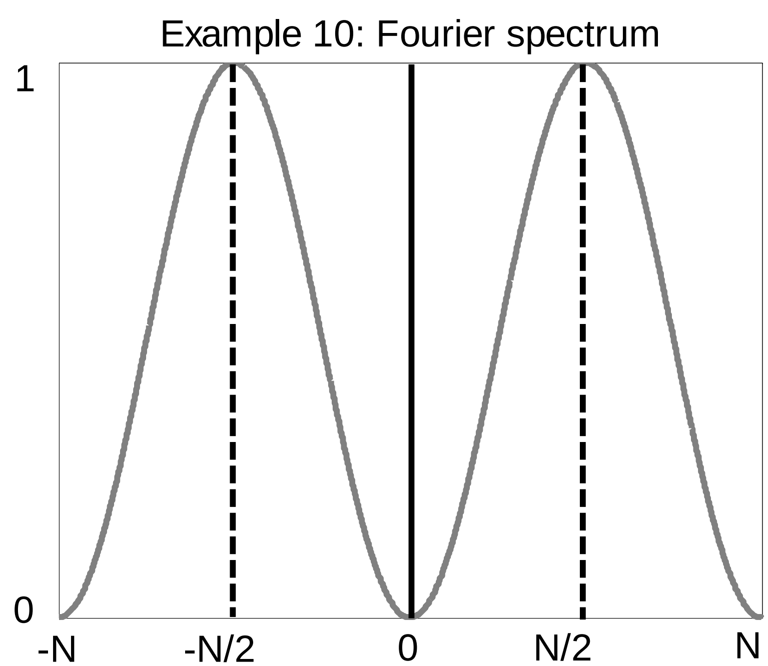
Consider a mask $g = [\, g(-1), \quad g(0), \quad g(1)\,]$, **where** $\sum g = 0$

(this condition guarantees a highpass filter in Fourier space ... later)

Example 10: **When** $g = [\, -\frac{1}{4}, \quad \frac{1}{2}, \quad -\frac{1}{4}\,]$, **we can show that**

$$\hat{G}(u) = \sin^2(\pi u/N),\ \hat{G}(0) = 0 \text{ and } \left|\hat{G}(N/2)\right| = \left|\hat{G}(-N/2)\right| = 1$$

(verify)



This implies a HIGHPASS FILTER

Continuous convolution in two dimensions

Definition: Consider two functions, $f(x,y)$ and $g(x,y)$, then

$$(f\bigstar g)(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\, g(x-\alpha, y-\beta)\, d\alpha\, d\beta$$

Convolution theorem in two dimensions

$$(f\bigstar g)(x,y) \Leftrightarrow F(u,v)\, G(u,v)$$

(verify)

Discrete convolution in two dimensions

Definition: Consider two matrices, $f(x,y)$ and $g(x,y)$, each with M rows and N columns, then

$$(f\bigstar g)(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\, g(x-m, y-n)$$

We can now again consider two-dimensional convolution masks, where the response is as follows for the linear case,

$$R = w_1 z_1 + w_2 z_2 + \ldots + w_9 z_9$$

Example of a smoothing mask, that is a lowpass filter in frequency space

$\sum_i w_i = 1:$

$$\frac{1}{9} \times \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

Example of a sharpening mask, that is a highpass filter in frequency space

$\sum_i w_i = 0:$

$$\frac{1}{9} \times \begin{array}{|c|c|c|} \hline -1 & -1 & -1 \\ \hline -1 & 8 & -1 \\ \hline -1 & -1 & -1 \\ \hline \end{array}$$

Note that when the dimensions of an image is greater than 32×32, we prefer to apply a filter in Fourier space, rather than use a convolution mask in physical space, since the FFT-algorithm guarantees that this is the more efficient option.

CORRELATION

Continuous correlation in one dimension

Definition: The correlation of $f(x)$ and $g(x)$ is

$$(f \star g)(x) = \int_{-\infty}^{\infty} f^*(\alpha)\, g(x+\alpha)\, d\alpha$$

The one function shifts with respect to the other, and is not flipped first in a left-right direction, as is the case for convolution

Correlation theorems in two dimensions

$$(f \star g)(x,y) \Leftrightarrow F^*(u,v)\, G(u,v)$$ **(verify)**

$$(f \star f)(x,y) \Leftrightarrow |\, F(u,v)\,|^2$$ **(verify)**

Discrete correlation in two dimensions

Definition: Consider two matrices, $f(x,y)$ and $g(x,y)$, then

$$(f \star g)(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m,n)\, g(x+m, y+n)$$

Application of correlation: Template/prototype matching (G&W: p 891-894)

Simplest form of correlation between $f(x,y)$ and $w(x,y)$

$$c(x,y) = \sum_s \sum_t w(s,t)\, f(x+s, y+t)$$

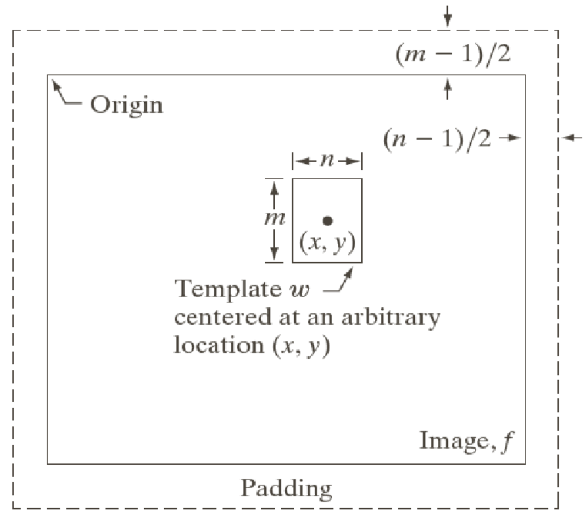


FIGURE 12.8 The mechanics of template matching.

For amplitude normalization: correlation coefficient

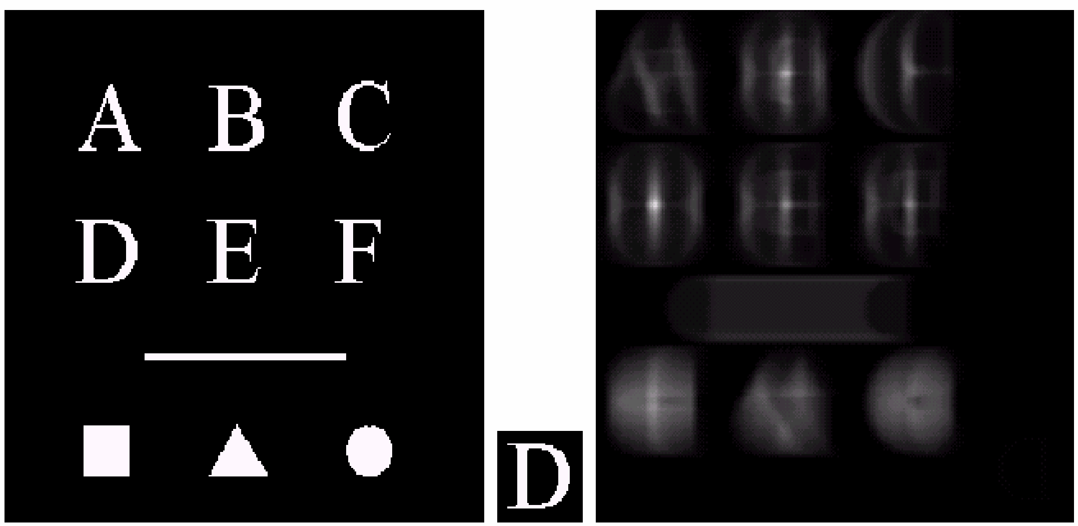
$$\gamma(x,y) = \frac{\sum_s \sum_t [w(s,t) - \overline{w}] \sum_s \sum_t \left[f(x+s,y+t) - \overline{f}(x+s,y+t)\right]}{\left\{\sum_s \sum_t [w(s,t) - \overline{w}]^2 \sum_s \sum_t \left[f(x+s,y+t) - \overline{f}(x+s,y+t)\right]^2\right\}^{\frac{1}{2}}}$$

It can be shown that $\gamma(x, y) \in [-1, 1]$.

Scale and rotation normalization can be difficult though. Why?

Remember that this can be achieved in Fourier space as well.

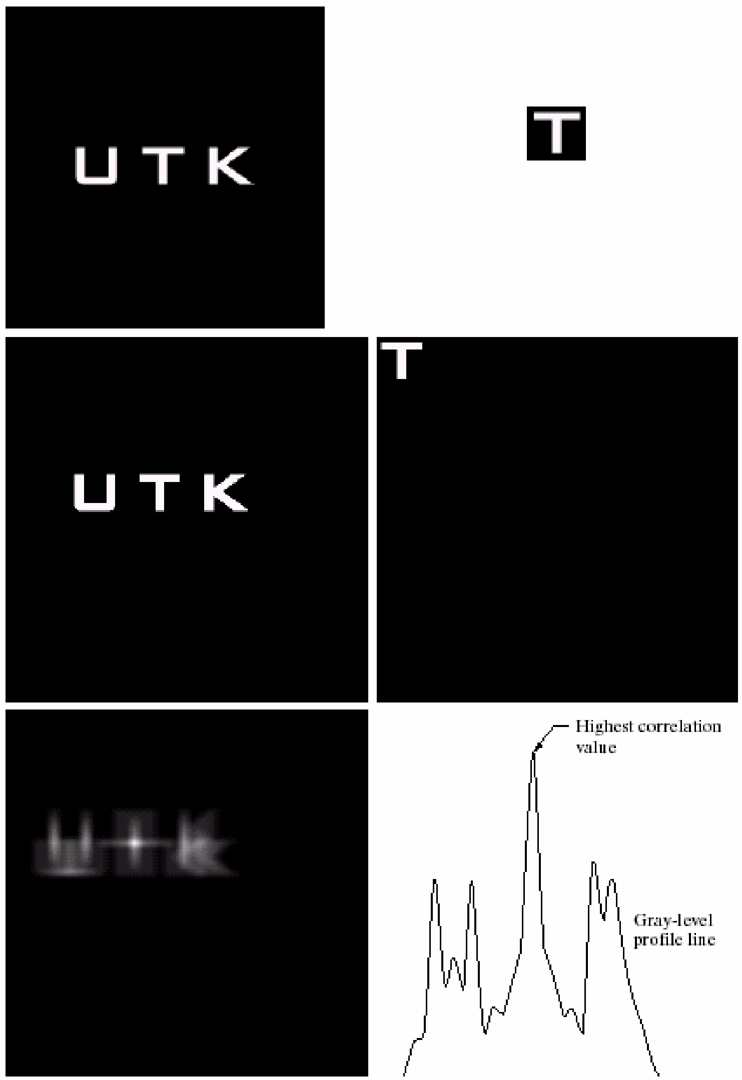
Example 12.2: (G&W: Ver 2) **Object matching via the correlation coefficient**



a b c

FIGURE 12.9
(a) Image.
(b) Subimage.
(c) Correlation coefficient of (a) and (b). Note that the highest (brighter) point in (c) occurs when subimage (b) is coincident with the letter "D" in (a).

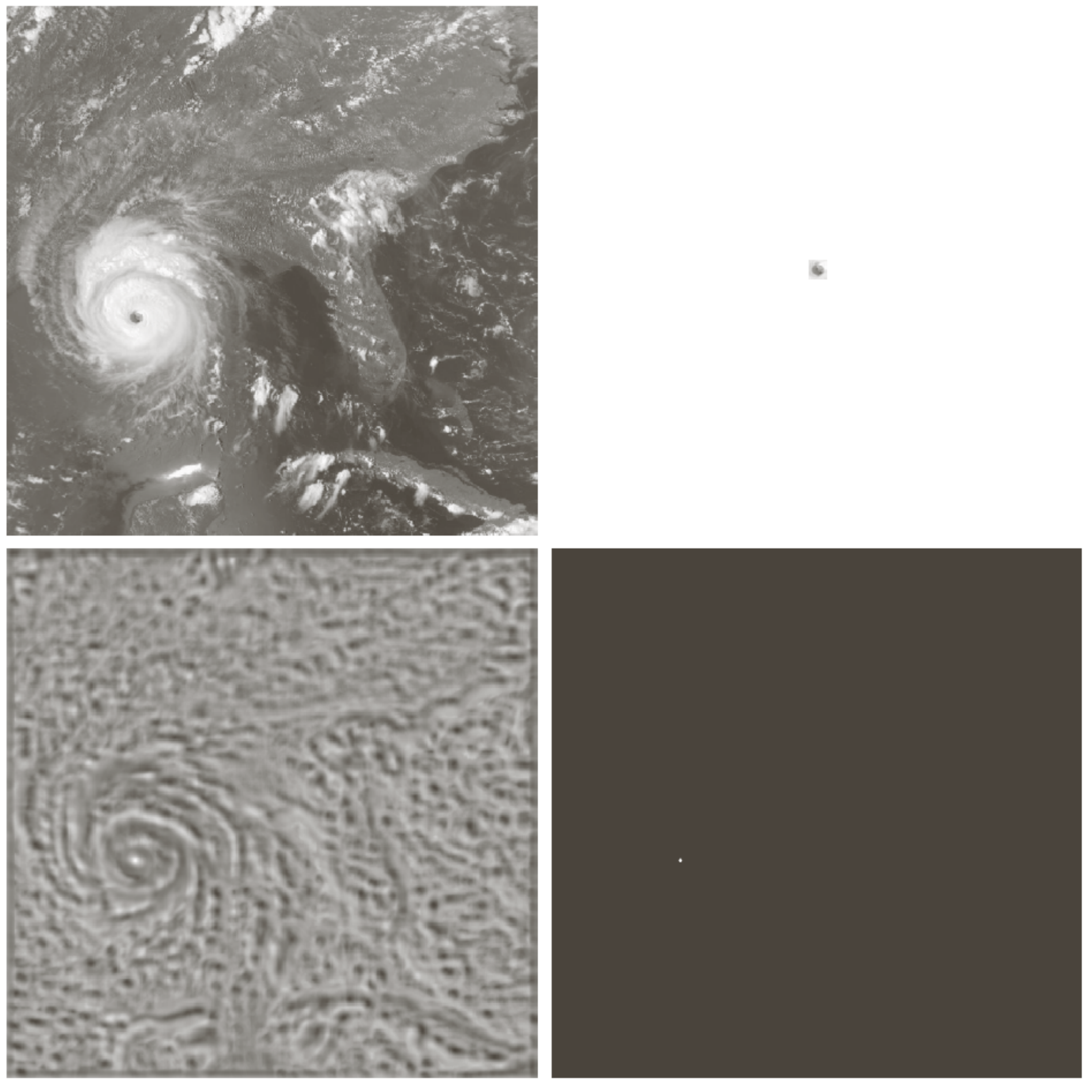
Example 4.11: (G&W: Ver 2) Image correlation



a b
c d
e f

FIGURE 4.41
(a) Image.
(b) Template.
(c) and
(d) Padded images.
(e) Correlation function displayed as an image.
(f) Horizontal profile line through the highest value in (e), showing the point at which the best match took place.

Example 12.2: (G&W: Ver 3) (page 893) Matching by correlation



a b
c d

FIGURE 12.9 (a) Satellite image of Hurricane Andrew, taken on August 24, 1992. (b) Template of the eye of the storm. (c) Correlation coefficient shown as an image (note the brightest point). (d) Location of the best match. This point is a single pixel, but its size was enlarged to make it easier to see. (Original image courtesy of NOAA.)