11.4 Use of principal components for description

- **Applicable to boundaries and regions**
- **Can also describe sets of images that were registered differently, for example the three component images of a color RGB image...**
- **Treat the three images as a unit by expressing each group of corresponding pixels as a vector...**

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- **When we have n registered images...**

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- **When $K = M$(rows) $\times$ N(columns), the <u>mean vector</u> of the population is defined as**

$$\mathbf{m_x} = E\{\mathbf{x}\} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{x}_k$$

• When $K = M$(rows)$\times N$(columns), the covariance matrix of the population is defined as

$$\begin{aligned}\mathbf{C_x} &= E\{(\mathbf{x}-\mathbf{m_x})(\mathbf{x}-\mathbf{m_x})^T\}\\ &= \frac{1}{K}\sum_{k=1}^{K}(\mathbf{x}_k-\mathbf{m_x})(\mathbf{x}_k-\mathbf{m_x})^T\\ &= \frac{1}{K}\sum_{k=1}^{K}\mathbf{x}_k\mathbf{x}_k^T-\mathbf{m_x}\mathbf{m_x}^T\end{aligned}$$

• Note that $\mathrm{C}_{\mathbf{x}}$ is an $n \times n$ matrix
• Element c_{ii} of $\mathrm{C}_{\mathbf{x}}$ is the variance of x_i
• Element c_{ij} of $\mathrm{C}_{\mathbf{x}}$ is the covariance between x_i and x_j
• The matrix $\mathrm{C}_{\mathbf{x}}$ is real and symmetric
• If x_i and x_j are uncorrelated, their covariance is zero, that is $c_{ij} = c_{ji} = 0$

Example 11.14: Mean vector and covariance matrix

Consider the four vectors $\mathbf{x}_1 = (0,0,0)^T$, $\mathbf{x}_2 = (1,0,0)^T$, $\mathbf{x}_3 = (1,1,0)^T$, and $\mathbf{x}_4 = (1,0,1)^T$, then

$$\mathbf{m_x} = \frac{1}{4}\begin{bmatrix}3\\1\\1\end{bmatrix}$$

$$\mathbf{C_x} = \frac{1}{16}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The three components have the same variance
Elements x_1 and x_2, and x_1 and x_3 are positively correlated
Elements x_2 and x_3 are negatively correlated

- **Since $\mathrm{C_x}$ is real and symmetric, we can always find a set of n orthonormal eigenvectors**
- **Let e_i and $\lambda_i,\ i = 1, \ldots, n$ be the eigenvectors and corresponding eigenvalues of $\mathrm{C_x}$ arranged in descending order so that $\lambda_j \geq \lambda_{j+1}$ for $j = 1, 2, \ldots, n-1$**
- **Let A be a matrix whose rows are formed from the eigenvectors of $\mathrm{C_x}$, ordered so that the first row is the eigenvector corresponding to the largest eigenvalue and the last row is the eigenvector corresponding to the smallest eigenvalue**
- **Suppose that A is a transformation matrix that maps the x's into vectors denoted by y's as follows:** $\boxed{\mathbf{y} = \mathbf{A}(\mathbf{x} - \mathbf{m_x})}$
- **This expression is called the <u>Hotelling transform</u> and has some interesting and useful properties...**

• It is possible to prove the following:

$$\mathbf{m_y} = E\{\mathbf{y}\} = \mathbf{0}$$

$$\mathbf{C_y} = \mathbf{A}\mathbf{C_x}\mathbf{A}^T$$

• Furthermore, $\mathrm{C}_\mathbf{y}$ is a diagonal matrix whose elements along the main diagonal are the eigenvalues of $\mathrm{C}_\mathbf{x}$, that is

$$\mathbf{C_y} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

• Note that the elements of the y vectors are uncorrelated

• Also, $\mathrm{C}_\mathbf{x}$ and $\mathrm{C}_\mathbf{y}$ have the same eigenvalues; the eigenvectors of $\mathrm{C}_\mathbf{y}$ are in the direction of the main axes

Inverse Hotelling transform: $\boxed{\mathbf{x} = \mathbf{A}^T\mathbf{y} + \mathbf{m_x}}$

• Suppose that instead of using all the eigenvectors of $\mathrm{C}_\mathbf{x}$ we form matrix A_k from the k eigenvectors corresponding to the k largest eigenvalues, yielding a transformation matrix of order $k \times n$

• The y vectors will then be k dimensional and the reconstruction will no longer be exact

- **The vector reconstructed by using $\mathbf{A}_k$ is**

$$\hat{\mathbf{x}} = \mathbf{A}_k^T \mathbf{y} + \mathbf{m}_{\mathbf{x}}$$

- **It can be shown that the mean square error between x and $\hat{\mathrm{x}}$ is given by the expression**

$$\begin{aligned} e_{\mathbf{ms}} &= \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{k} \lambda_j \\ &= \sum_{j=k+1}^{n} \lambda_j \end{aligned}$$

- **The first line indicates that the error is zero if $k = n$, that is if all the eigenvectors are used in the transformation**
- **Note that the error can be minimized by selecting the k eigenvectors associated with the largest eigenvalues**
- **The Hotelling transform is optimal in the sense that it minimizes the mean square error between x and $\hat{\mathrm{x}}$**
- **Due to this idea of using the eigenvectors corresponding with the largest eigenvalues, the Hotelling transform also is known as the <u>principal components</u> transform**

Example 11.15: Using principal components for image description

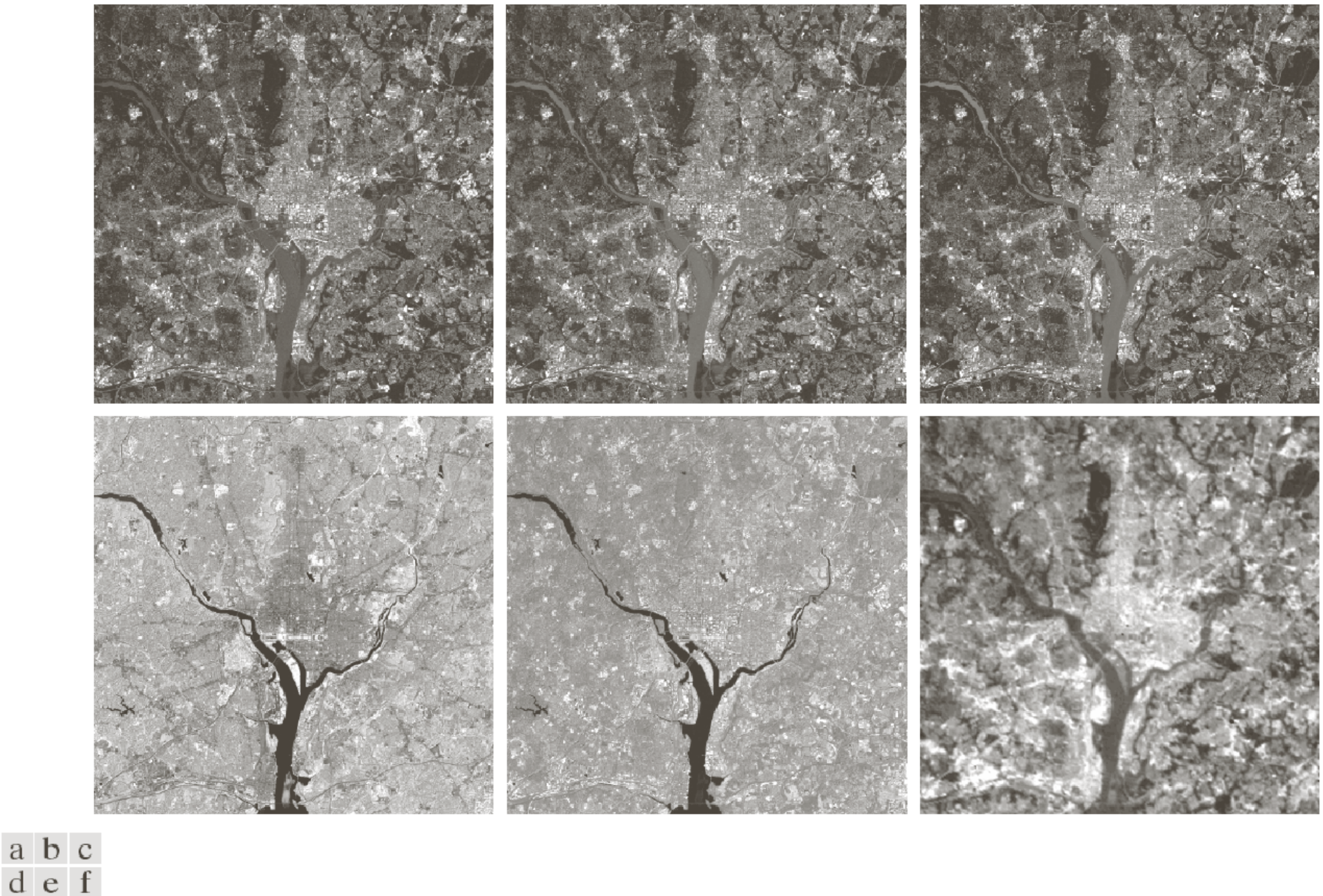


FIGURE 11.38 Multispectral images in the (a) visible blue, (b) visible green, (c) visible red, (d) near infrared, (e) middle infrared, and (f) thermal infrared bands. (Images courtesy of NASA.)

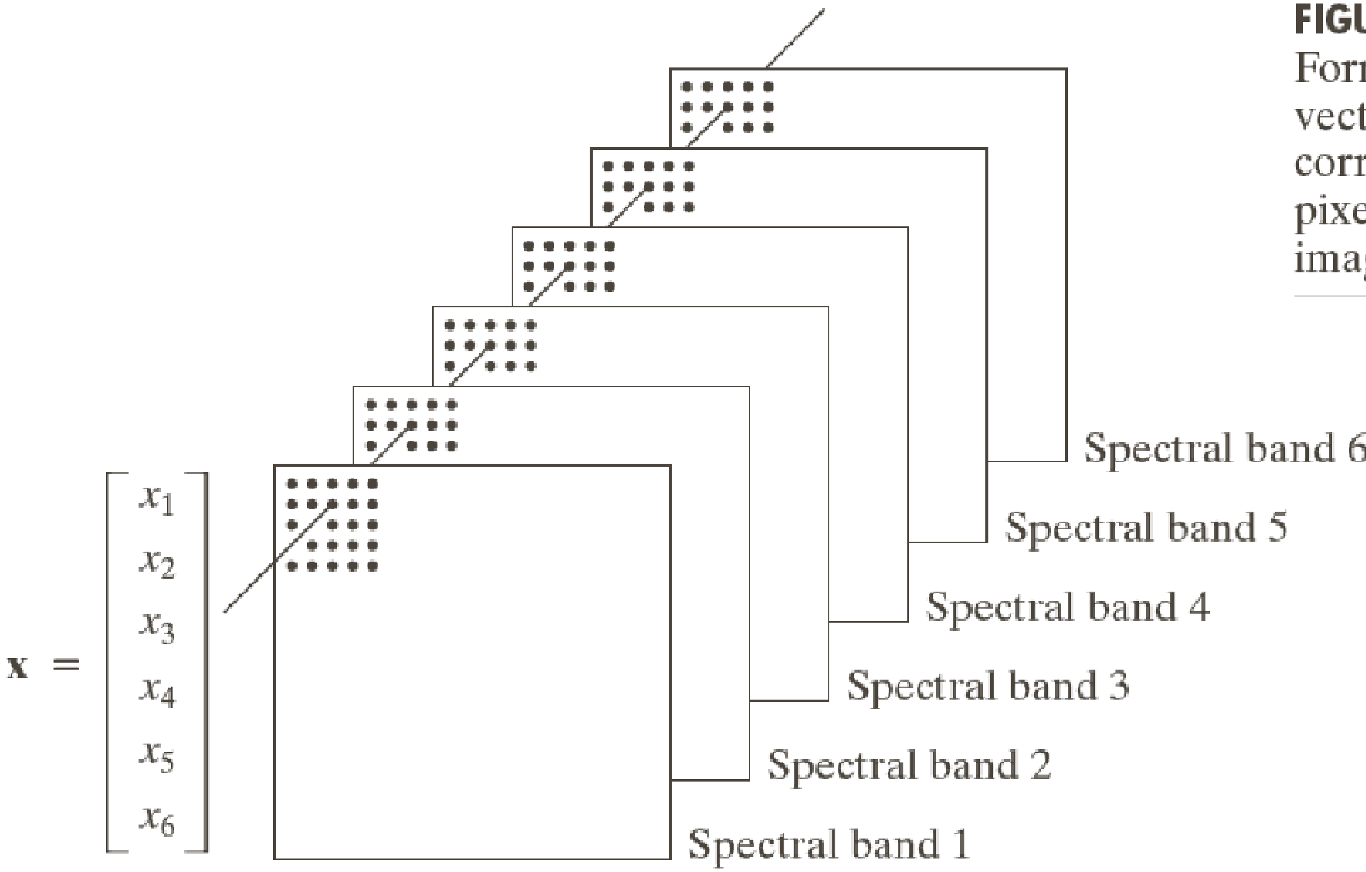
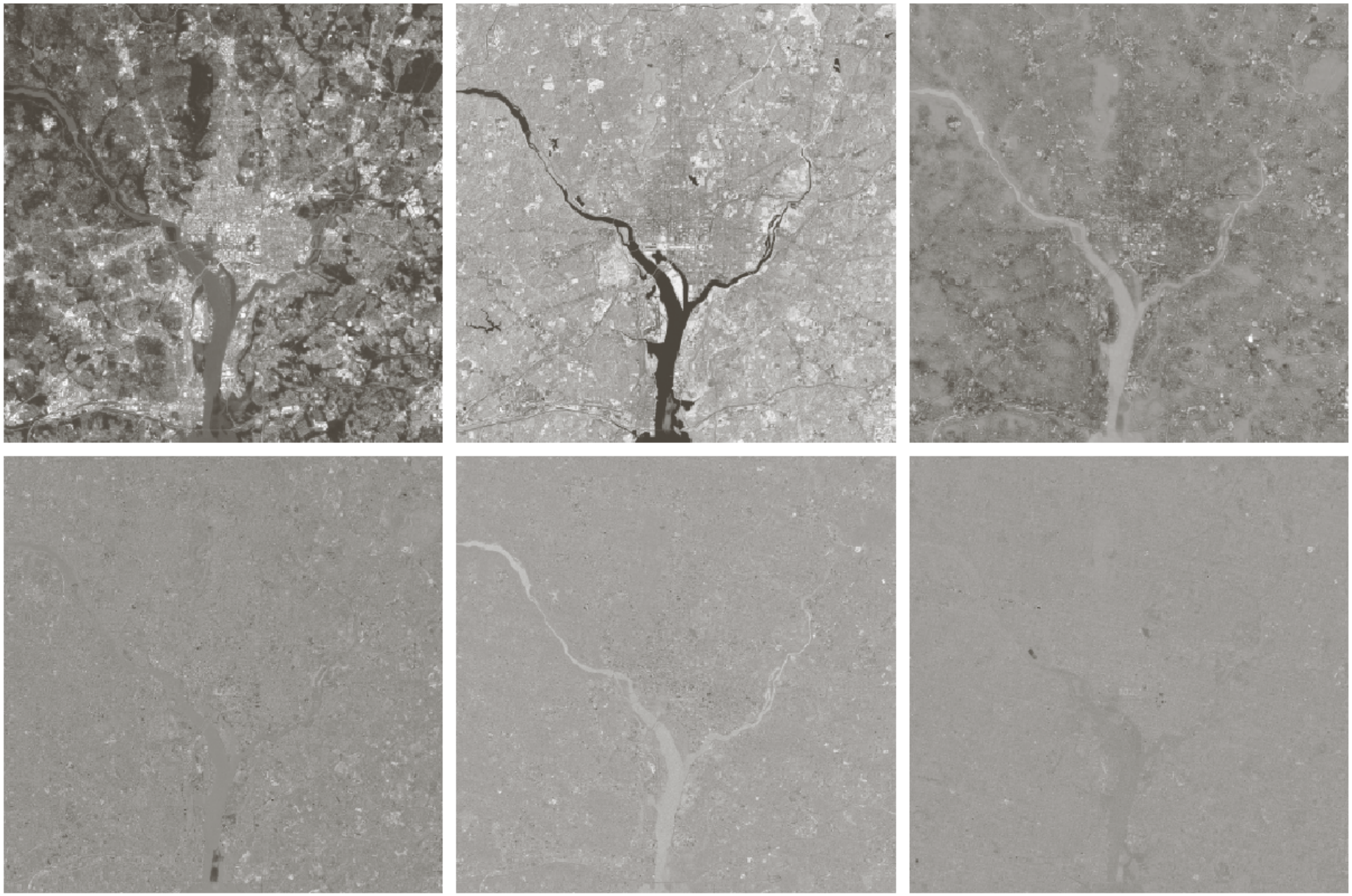


FIGURE 11.39 Formation of a vector from corresponding pixels in six images.

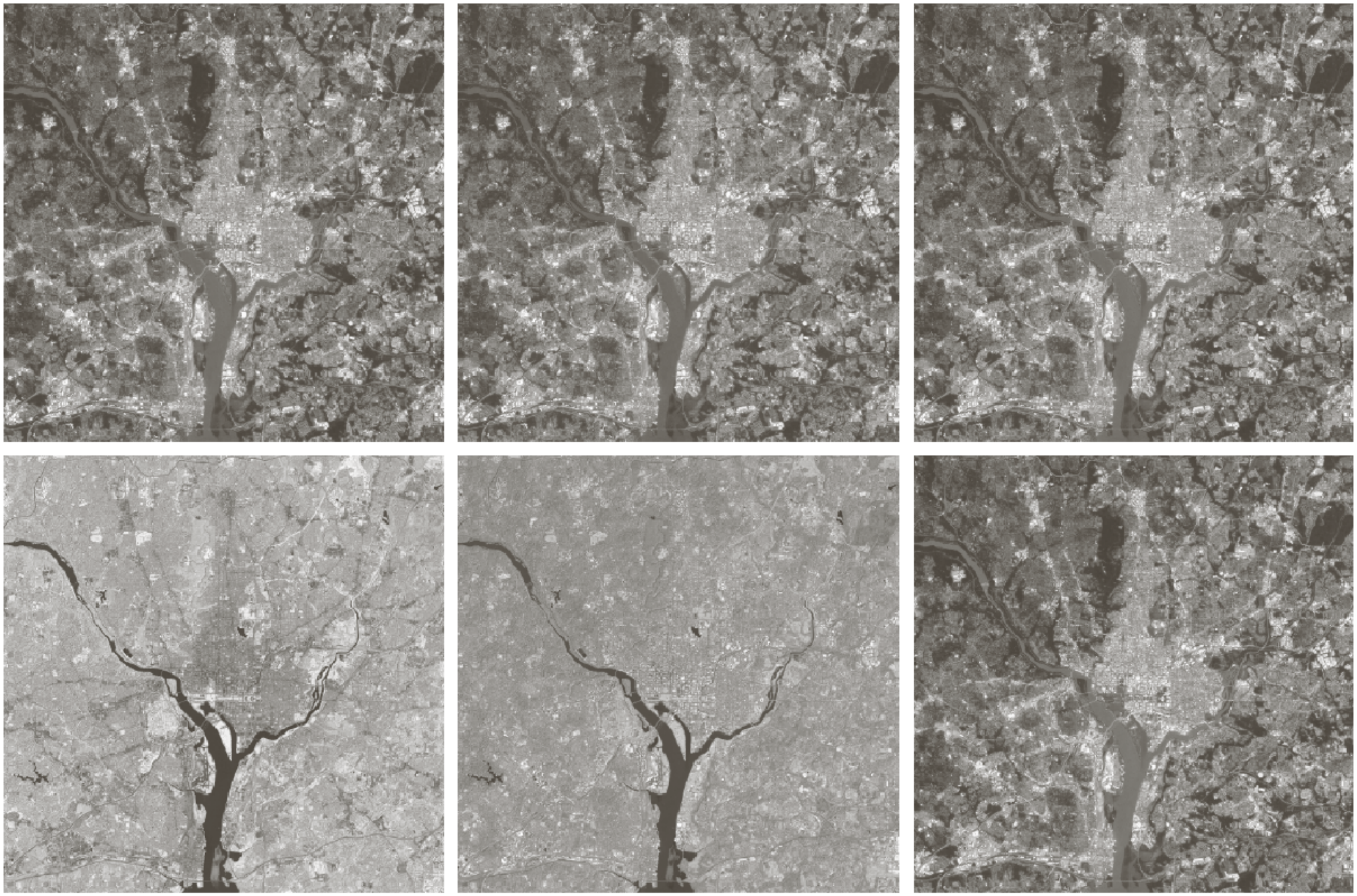
λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
10344	2966	1401	203	94	31

TABLE 11.6 Eigenvalues of the covariance matrices obtained from the images in Fig. 11.38.



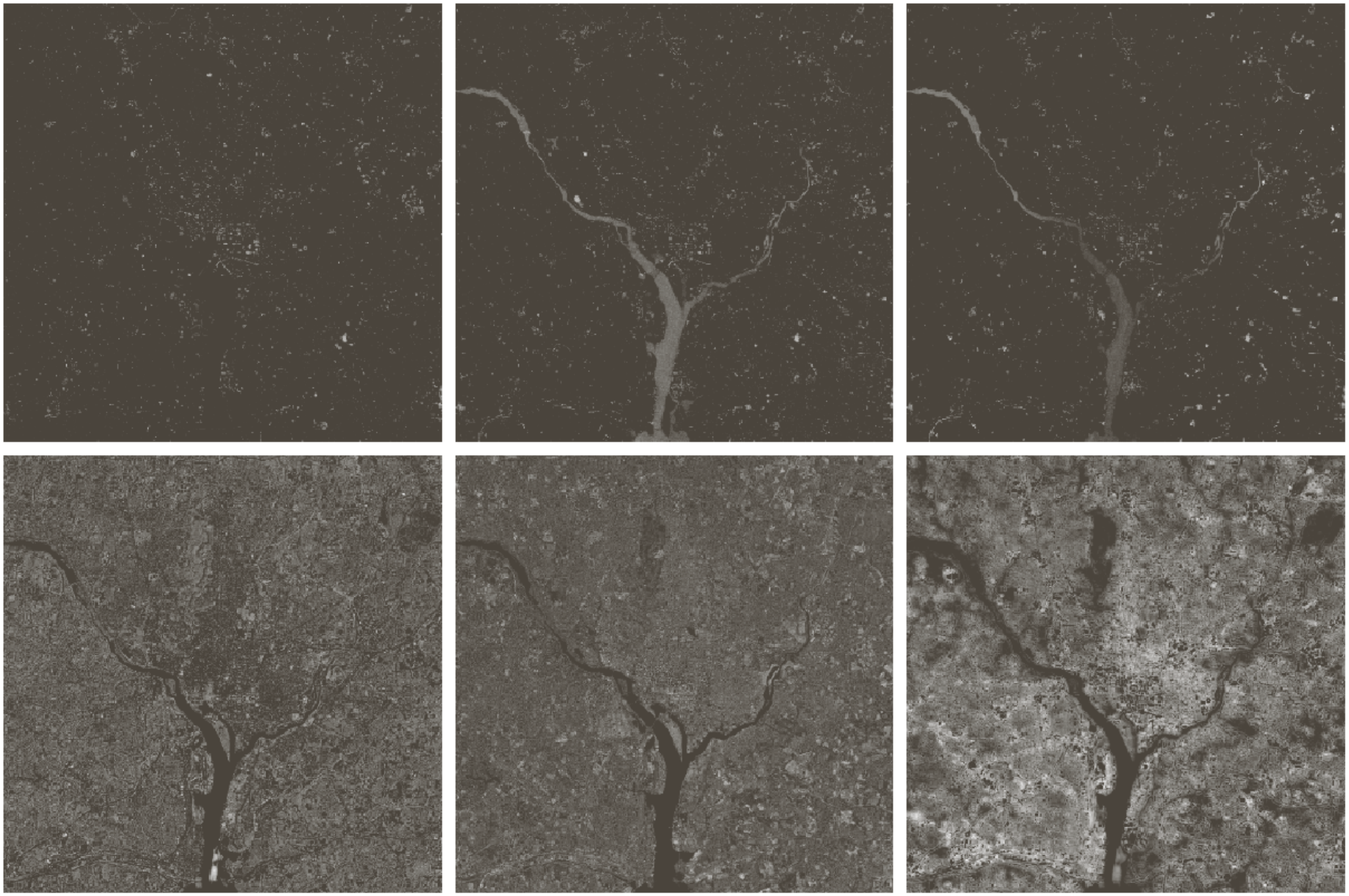
a	b	c
d	e	f

FIGURE 11.40 The six principal component images obtained from vectors computed using Eq. (11.4-6). Vectors are converted to images by applying Fig. 11.39 in reverse.



a	b	c
d	e	f

FIGURE 11.41 Multispectral images reconstructed using only the two principal component images corresponding to the two principal component images with the largest eigenvalues (variance). Compare these images with the originals in Fig. 11.38.



a	b	c
d	e	f

FIGURE 11.42 Differences between the original and reconstructed images. All difference images were enhanced by scaling them to the full [0, 255] range to facilitate visual analysis.

Example 11.16: Using PCs for size, translation, and rotation normalization

$$\mathbf{m}_\mathbf{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\mathbf{C}_\mathbf{x} = \begin{bmatrix} 3.333 & 2.00 \\ 2.00 & 3.333 \end{bmatrix}$$

$$\mathbf{e}_1 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$$

$$\mathbf{e}_2 = \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix}$$

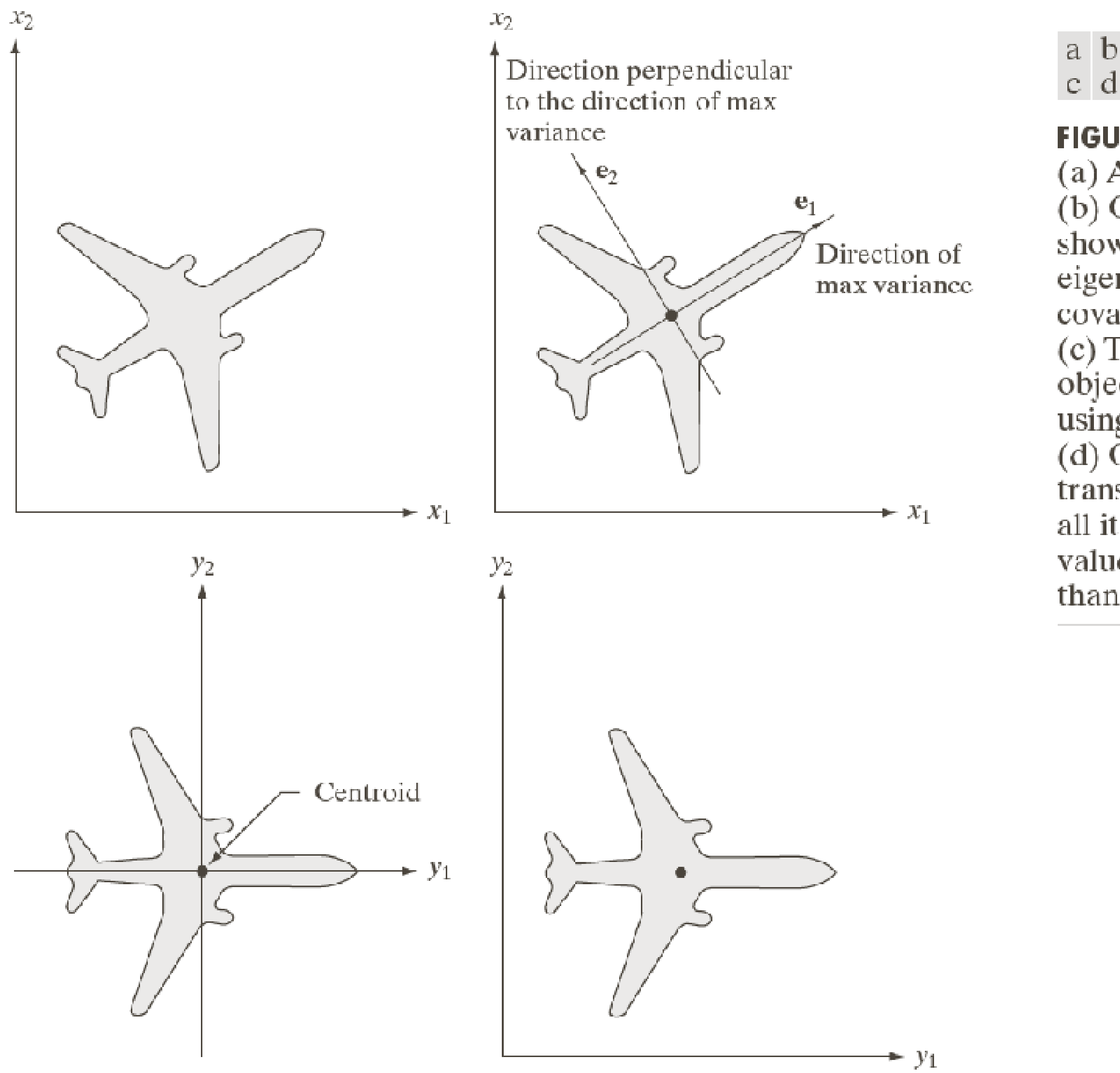
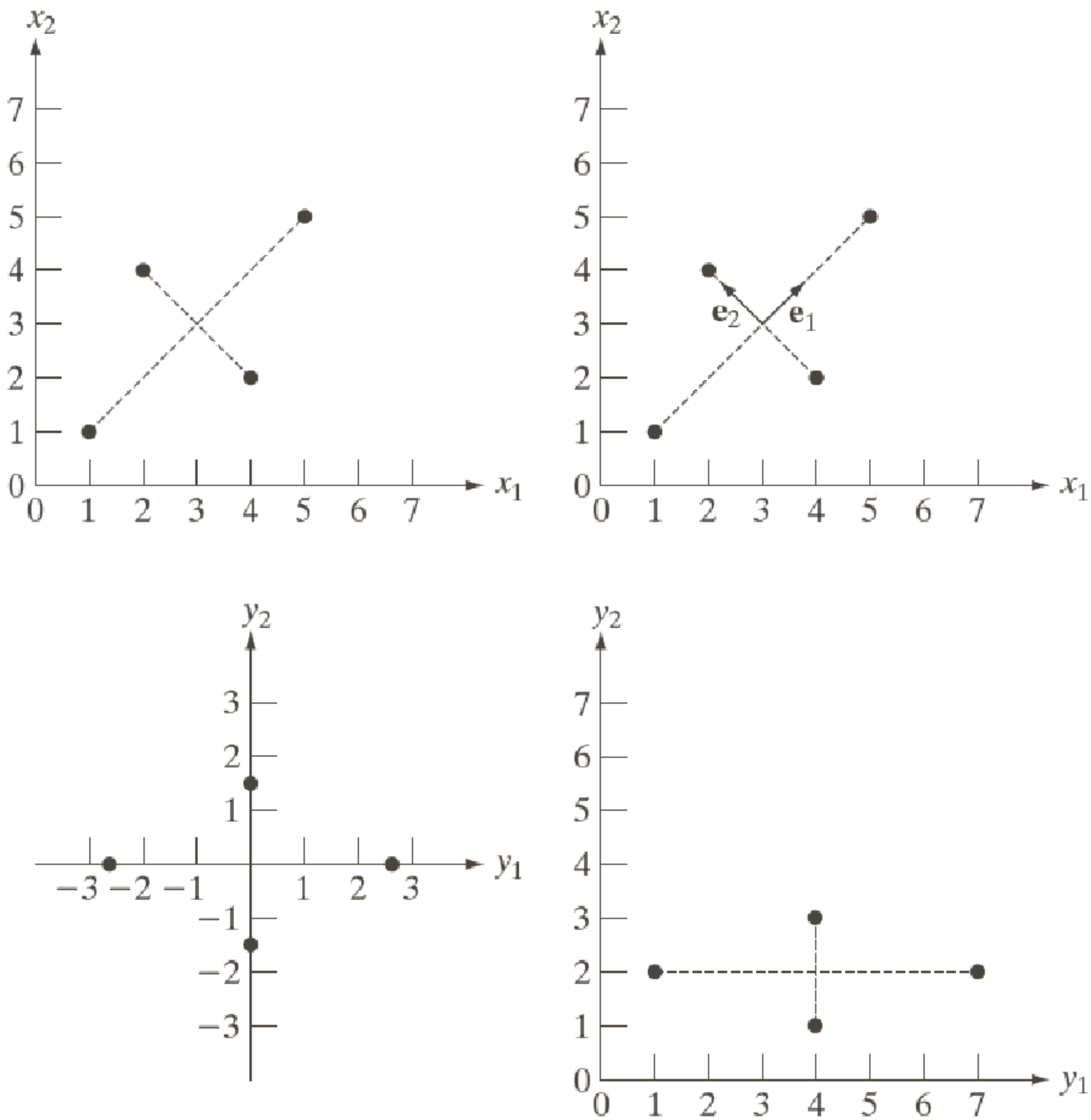


FIGURE 11.43 (a) An object. (b) Object showing eigenvectors of its covariance matrix. (c) Transformed object, obtained using Eq. (11.4-6). (d) Object translated so that all its coordinate values are greater than 0.



a b
c d

FIGURE 11.44 A manual example. (a) Original points. (b) Eigenvectors of the covariance matrix of the points in (a). (c) Transformed points obtained using Eq. (11.4-6). (d) Points from (c), rounded and translated so that all coordinate values are integers greater than 0. The dashed lines are included to facilitate viewing. They are not part of the data.