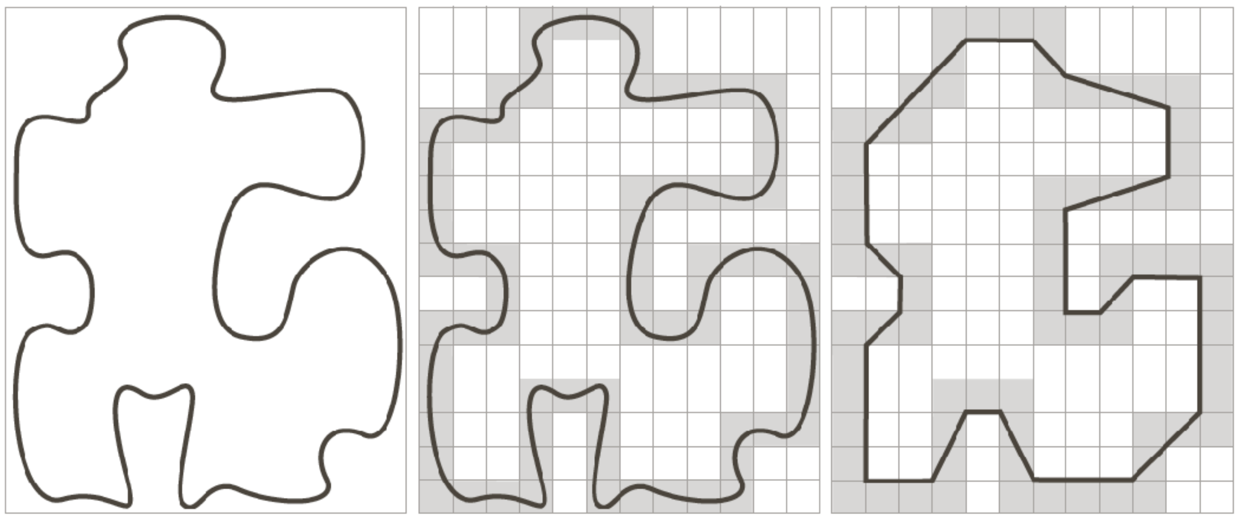
11.1.3 Polygonal approximations using minimum-perimeter polygons (MPPs)

Cellular complex $\equiv$ set of cells enclosing digital boundary

Imagine the boundary as a “rubber band” and allow it to shrink...



a b c

FIGURE 11.6 (a) An object boundary (black curve). (b) Boundary enclosed by cells (in gray). (c) Minimum-perimeter polygon obtained by allowing the boundary to shrink. The vertices of the polygon are created by the corners of the inner and outer walls of the gray region.

The maximum error per grid cell is $\sqrt{2}d$, where d is the dimension of a grid cell

MPP algorithm (READ)

Example 11.3: Applying the MPP algorithm

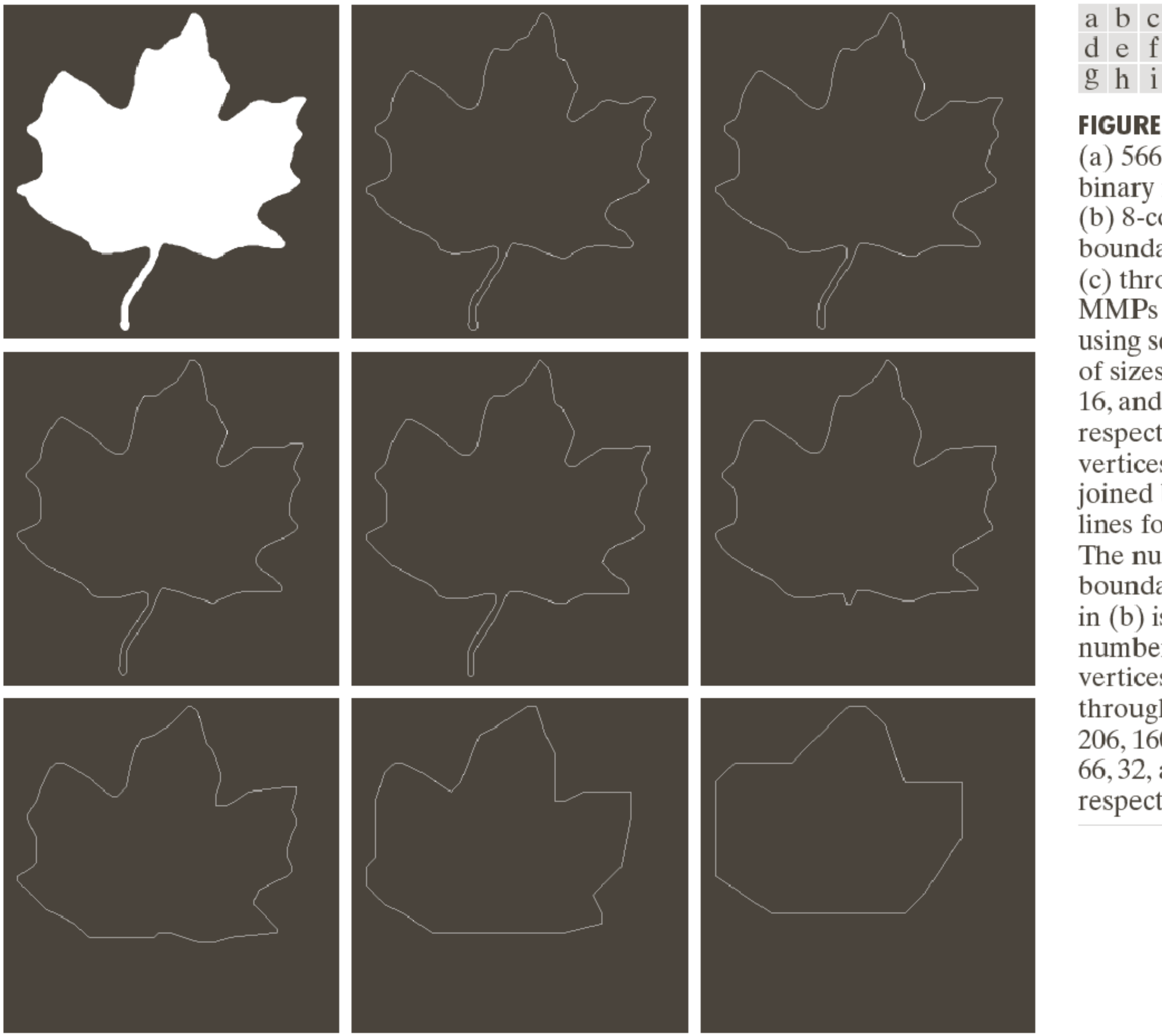


FIGURE 11.8 (a) 566 × 566 binary image. (b) 8-connected boundary. (c) through (i), MMPs obtained using square cells of sizes 2, 3, 4, 6, 8, 16, and 32, respectively (the vertices were joined by straight lines for display). The number of boundary points in (b) is 1900. The numbers of vertices in (c) through (i) are 206, 160, 127, 92, 66, 32, and 13, respectively.

11.1.4 Other polygonal approximation approaches

Merging techniques

(1) Consider an arbitrary point on the boundary

(2) Consider the next point and fit a line through these two points: $E = 0$ (least squares error is zero)

(3) Now consider the next point as well, and fit a line through all three these points using a least squares approximation. Calculate E

(4) Repeat until $E > T$

(5) Store a and b of $y = ax + b$, and set $E = 0$

(6) Find the following line and repeat until all the edge pixels were considered

(7) Calculate the vertices of the polygon, that is where the lines intersect

Problems: Vertices do not always correspond to actual corners in the boundary: a new line is started after we have already turned, that is T is exceeded too late

Splitting techniques

- Joint the two furthest points on the boundary $\longrightarrow$ line ab
- Obtain a point on the upper segment, that is c and a point on the lower segment, that is d, such that the perpendicular distance from these points to ab is as large as possible
- Now obtain a polygon by joining c and d with a and b
- Repeat until the perpendicular distance is less than some predefined fraction of ab

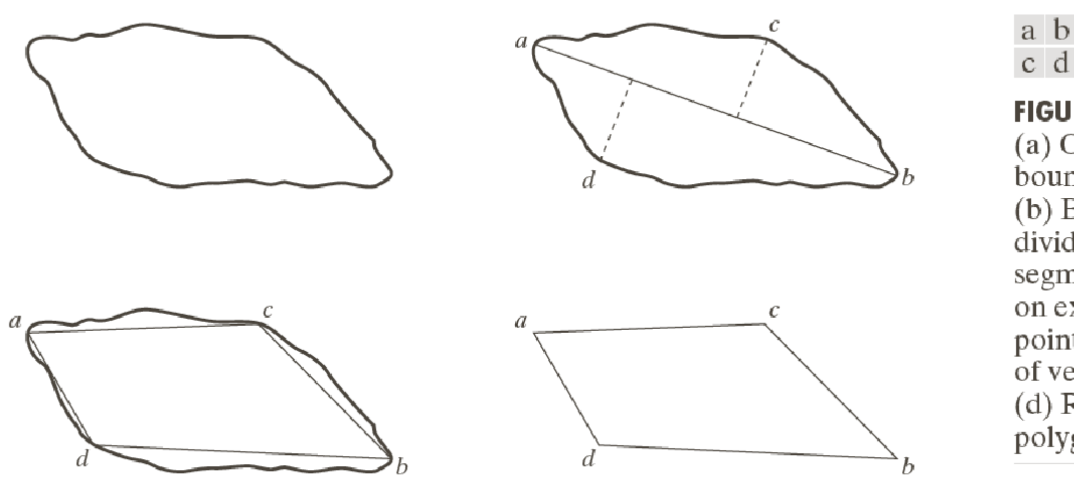


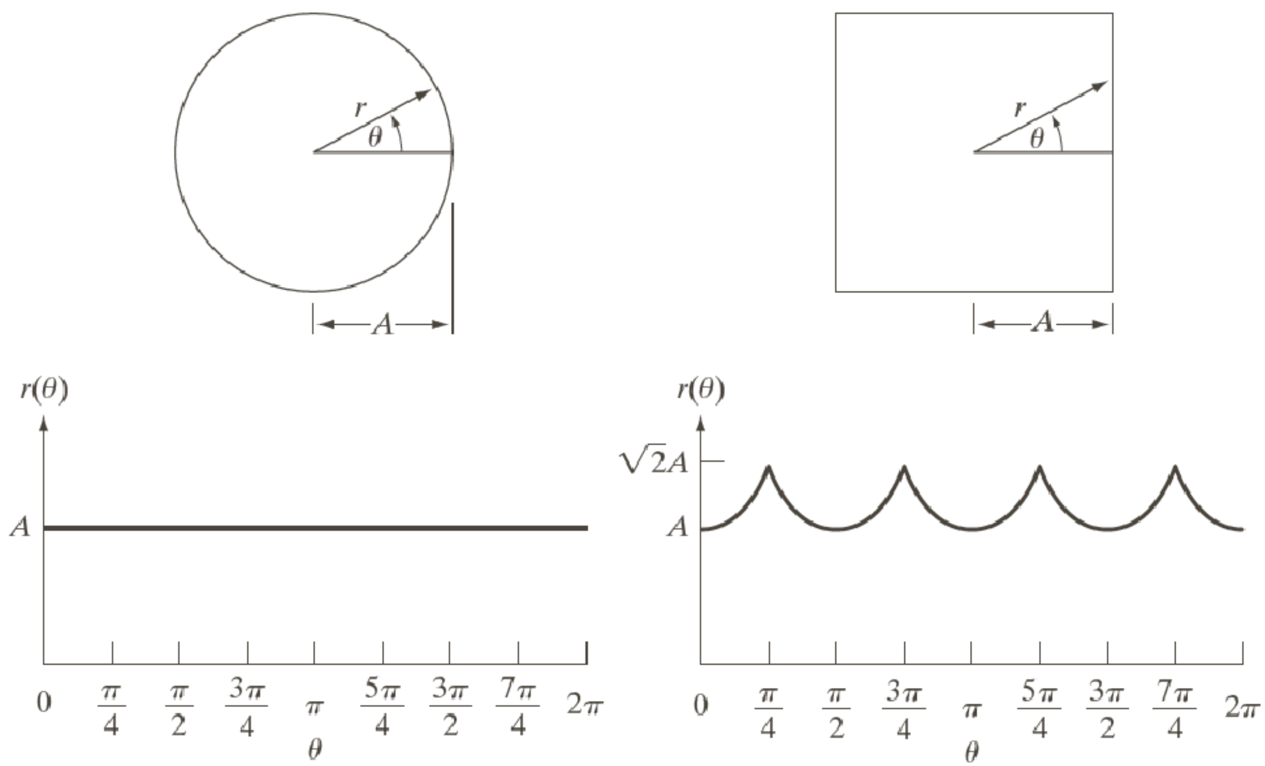
FIGURE 11.9 (a) Original boundary. (b) Boundary divided into segments based on extreme points. (c) Joining of vertices. (d) Resulting polygon.

11.1.5 “Signatures”

- **1-D representation of boundary: generated in various ways**

Simplest approach: plot $r(\theta)$

- **r: distance from centroid of boundary to boundary point**
- **θ: angle with the positive x-axis**



a b

FIGURE 11.10 Distance-versus-angle signatures. In (a) $r(\theta)$ is constant. In (b), the signature consists of repetitions of the pattern $r(\theta) = A \sec\theta$ for $0 \leq \theta \leq \pi/4$ and $r(\theta) = A \csc\theta$ for $\pi/4 < \theta \leq \pi/2$.

Translation invariant, but not rotation or scale invariant

<u>Normalization for rotation</u>

(1) Choose the starting point as the furthest point from the centroid OR

(2) Choose the starting point as the point on the major axis that is the furthest from the centroid

<u>Normalization for scale</u> Note: $\uparrow$ scale $\Rightarrow \uparrow$ amplitude of signature

(1) Scale signature between 0 and 1 Problem: sensitive to noise

(2) Divide each sample by its variance - assuming it is not zero

<u>Alternative approach:</u> plot $\Phi(\theta)$

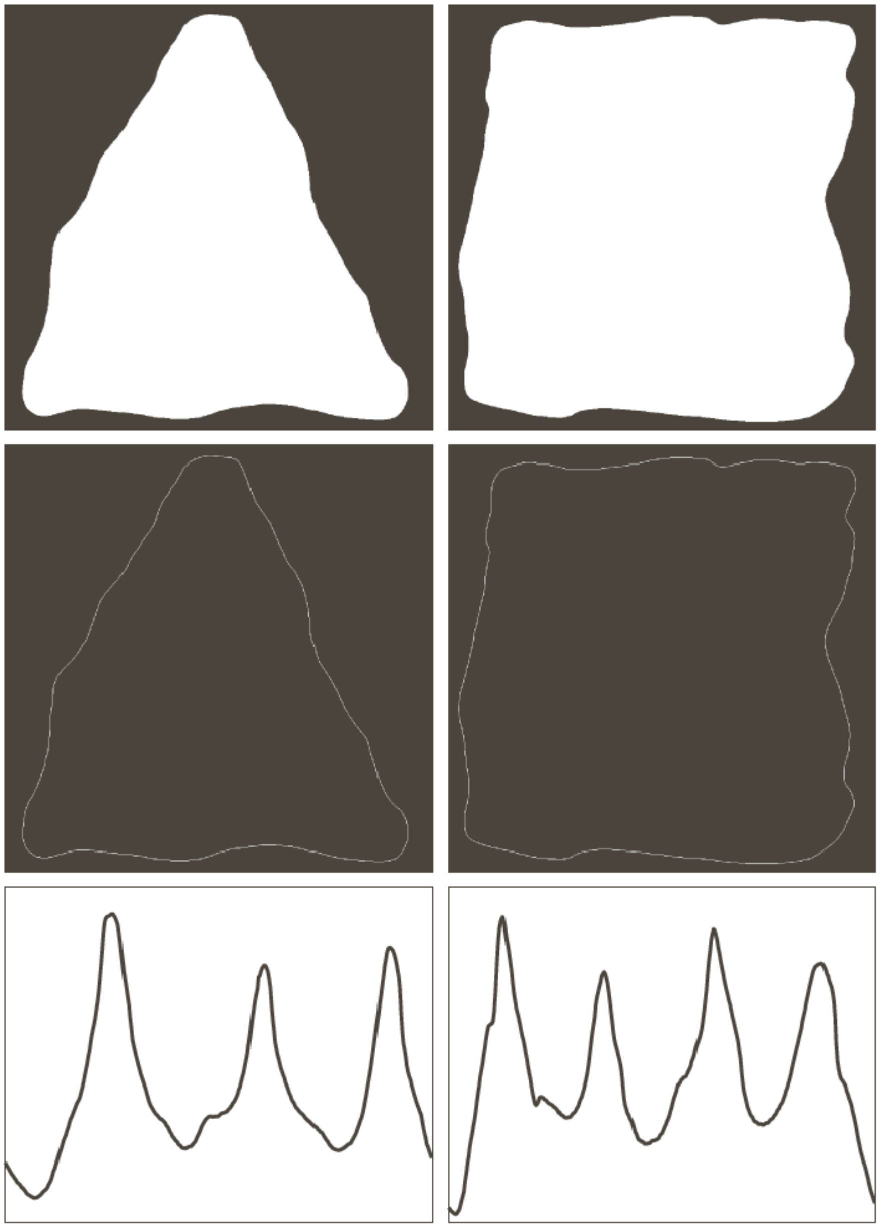
- Φ: angle between the line tangent to the boundary and a reference line
- θ: angle with the positive x-axis

$\Phi(\theta)$ carry information about basic shape characteristics

<u>Alternative approach:</u> use the so-called slope density function as a signature, that is a histogram of the tangent-angle values

- Respond strongly to sections of the boundary with constant tangent angles (straight or nearly straight segments)
- Deep valleys in sections producing rapidly varying angles (corners or other sharp inflections)

Example 11.4: Signatures of two simple opjects



a	b
c	d
e	f

FIGURE 11.11 Two binary regions, their external boundaries, and their corresponding $r(\theta)$ signatures. The horizontal axes in (e) and (f) correspond to angles from 0° to 360°, in increments of 1°.

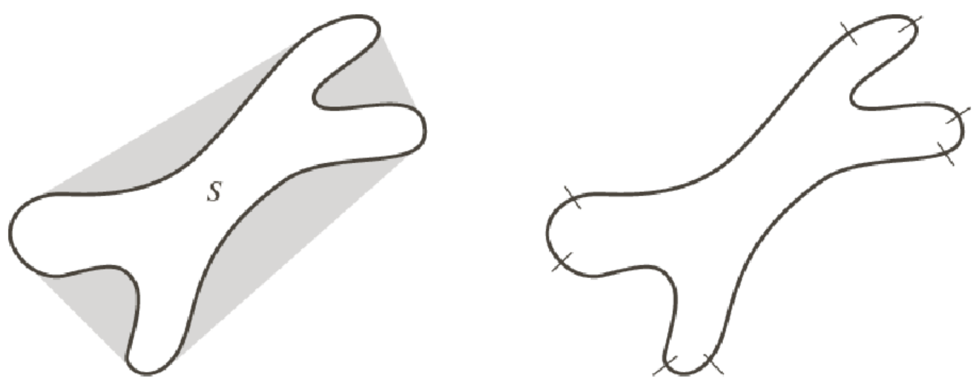
11.1.6 Boundary segments

Boundary segments are usually easier to describe than the boundary as a whole

We need a robust decomposition: convex hull

A convex set (region) is a set (region) in which any two elements (points) A and B in the set (region) can be joined by a line AB, so that each point on AB is part of the set (region)

The convex hull H of an arbitrary set (region) S is the smallest convex set (region) containing S



a b

FIGURE 11.12 (a) A region, S, and its convex deficiency (shaded). (b) Partitioned boundary.

Convex deficiency: $D = H - S$

The region boundary is partitioned by following the contour of S and marking the points at which a transition is made into or out of a component of the convex deficiency

The partitioning of irregular boundaries (that results from the digitization process or noise) usually leads to small meaningless components

We therefore smooth the boundary prior to partitioning:

(1) Use averaging mask on coordinates of boundary pixels OR

(2) Polygonal approximation prior to computation of convex deficiency

11.1.7 Skeletons: Already discussed

11.2 Boundary descriptors

11.2.1 Some simple descriptors

- **Length**
- **Diameter**
- **Major axis**
- **Minor axis**
- **Basic rectangle**
- **Eccentricity**

- **Curvature: rate of change of slope ...**

  ... vertex point p part of
  - **Convex segment**
  - **Concave segment**
  - **Straight segment**
  - **Corner point**

11.2.2 Shape numbers: Already discussed

11.2.3 Fourier descriptors

Suppose that a boundary is represented by K coordinate pairs in the xy-plane, (x_0, y_0), (x_1, y_1), $(x_2, y_2), \ldots, (x_{K-1}, y_{K-1})$

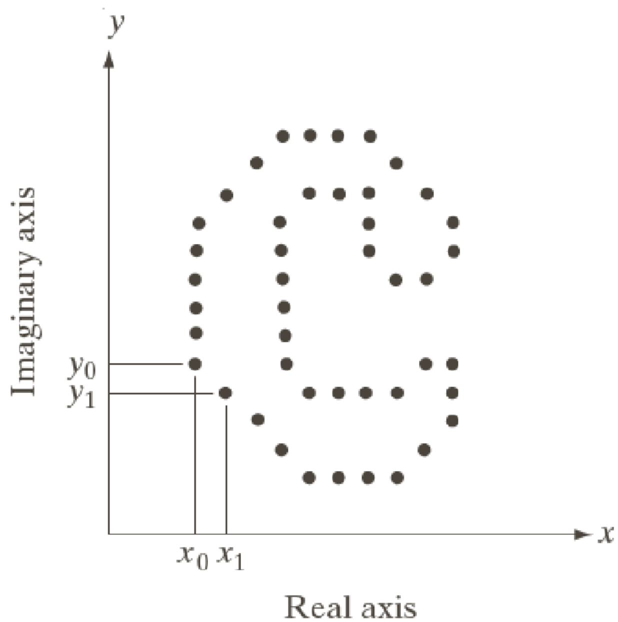


FIGURE 11.19 A digital boundary and its representation as a complex sequence. The points (x_0, y_0) and (x_1, y_1) shown are (arbitrarily) the first two points in the sequence.

When we traverse this boundary in an anti-clockwise direction the boundary can be represented as the sequence of coordinates $s(k) = [\,x(k),\, y(k)\,]$ for $k = 0, 1, 2, \ldots, K-1$
Each coordinate pair can be treated as a complex number so that $s(k) = x(k) + i\,y(k) \Rightarrow$ 2-D problem $\rightarrow$ 1-D problem

DFT of $s(k)$: $\hat{s}(u) = \sum_{k=0}^{K-1} s(k)\, e^{-2\pi iuk/K},\ u = 0, \ldots, K-1$ **(Fourier descriptors)**

DIFT of $\hat{s}(u)$: $s(k) = \frac{1}{K} \sum_{u=0}^{K-1} \hat{s}(u)\, e^{2\pi iuk/K},\ k = 0, \ldots, K-1$

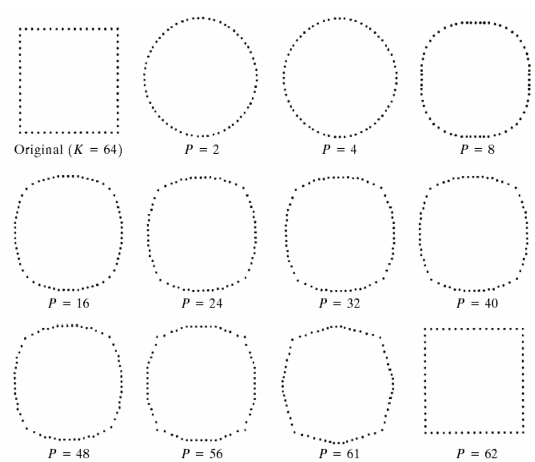
Suppose that we use only the first P coefficients...

$$\tilde{s}(k) = \frac{1}{P} \sum_{u=0}^{P-1} \hat{s}(u)\, e^{2\pi iuk/P},\ k = 0, \ldots, K-1$$

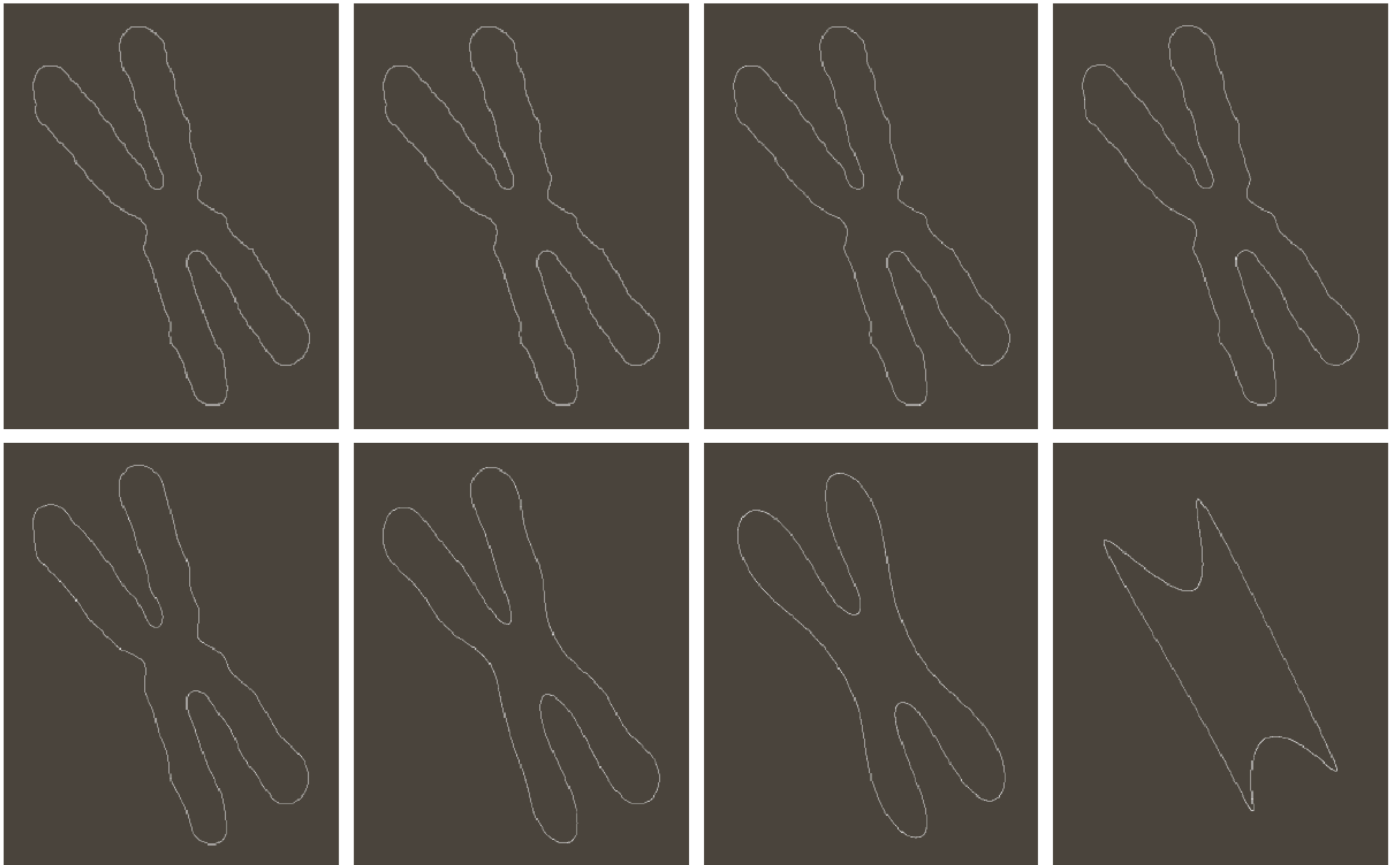
Note that the same number of points exist in the approximate boundary $\tilde{s}(k)$. Also note that the smaller P becomes, the more detail in the boundary is lost...

(2nd Ed)

FIGURE 11.14 Examples of reconstruction from Fourier descriptors. P is the number of Fourier coefficients used in the reconstruction of the boundary.



Example 11.7: Using Fourier descriptors



a	b	c	d
e	f	g	h

FIGURE 11.20 (a) Boundary of human chromosome (2868 points). (b)–(h) Boundaries reconstructed using 1434, 286, 144, 72, 36, 18, and 8 Fourier descriptors, respectively. These numbers are approximately 50%, 10%, 5%, 2.5%, 1.25%, 0.63%, and 0.28% of 2868, respectively.

Rotation invariance: **Consider the point in the complex plane,** $s(k) = x(k) + i\,y(k)$ **and another point** $r(k) = s(k)\,e^{i\theta}$

$$\begin{aligned} r(k) &= [x(k) + i\,y(k)]\,e^{i\theta} \\ &= [x(k) + i\,y(k)]\,[\cos\theta + i\sin\theta] \\ &= [x(k)\cos\theta - y(k)\sin\theta, x(k)\sin\theta + y(k)\cos\theta] \\ &= \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\substack{\textbf{Rotation matrix:} \\ \textbf{det} = \mathbf{1}}} \underbrace{\begin{pmatrix} x(k) \\ y(k) \end{pmatrix}}_{s(k)} \end{aligned}$$

This implies that $r(k)$ **is simply** $s(k)$ **that was rotated through an angle** θ **in the complex plane**

$$\hat{r}(u) = \sum_{k=0}^{K-1} \overbrace{s(k)\,e^{i\theta}}^{r(k)}\,e^{-2\pi iuk/K} = e^{i\theta}\sum_{k=0}^{K-1} s(k)\,e^{-2\pi iuk/K} = e^{i\theta}\,\hat{s}(u)$$

This implies that $|\hat{r}(u)| = |\hat{s}(u)|$ **and that rotation invariance can be achieved by comparing the Fourier spectra of the boundaries in question**

Translation invariance

Since the DC component of the Fourier transform, that is $\hat{s}(0)$ represents the average boundary coordinate in the complex plane, translation invariance can be achieved be simply setting $\hat{s}(0) = 0$. This will translate the boundary in such a way that the average boundary coordinate coincides with the origin of the complex plane

Scale invariance

When only two Fourier coefficients, that is $\hat{s}(-1)$ and $\hat{s}(1)$, are used, the boundary in the complex plane is approximated by an ellipse. The average amplitude or area of this ellipse is proportional to the scale of the boundary and can be used as a scaling factor

Invariance with respect to the starting point

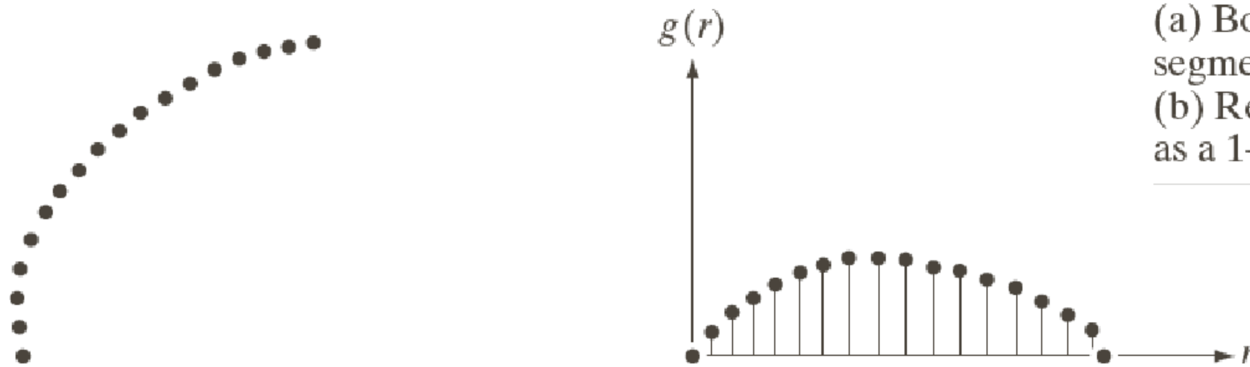
Invariance with respect to the starting point follows from the result that we proved earlier...

$$|\textbf{FT}\{\, s(k-k_0)\,\}| = |\textbf{FT}\{\, s(k)\,\}|$$

This implies that if the Fourier spectrum of the boundary coordinates is considered, rotation invariance and invariance with respect to the starting point are ensured

11.2.4 Statistical moments

Statistical moments can be used to describe shape of boundary segment; A boundary segment can be represented by a 1-D discrete function $g(r)$ ...



a b

FIGURE 11.21
(a) Boundary segment.
(b) Representation as a 1-D function.

The amplitude of g can now be treated as a discrete random variable v so that a histogram $p(v_i),\ i = 0, 1, \ldots, A-1$ is formed, where A is the number of amplitude increments

The nth moment of v about its mean is $\mu_n(v) = \sum_{i=0}^{A-1} (v_i - m)^n \, p(v_i)$, where $m = \sum_{i=0}^{A-1} v_i \, p(v_i)$ is the mean and μ_2 the variance

Generally, only the first few moments are required to differentiate between signatures of clearly distinct shapes

Alternatively, treat $g(r_i)$ as the probability of value r_i occuring, so that the moments are

$$\mu_n(r) = \sum_{i=0}^{K-1} (r_i - m)^n \, g(r_i),$$

where

$$m = \sum_{i=0}^{K-1} r_i \, g(r_i)$$

Here K is the number of points on the boundary, and $\mu_n(r)$ is directly related to the shape of $g(r)$:

Spread of the curve: $\mu_2(r)$

Symmetry with reference to the mean: $\mu_3(r)$