CHECK BOX 1.3

(a)

What would the final specific solution of the PDE be, if boundary condition [**1**] in Example 1.2 is replaced by

$$u(x, 0) = 10 \sin(3\pi x/L)\,?$$

The solution is shown in Figure 1.4.

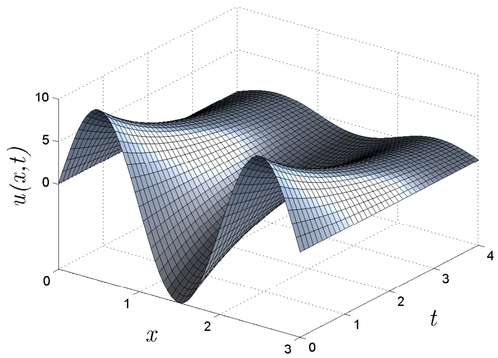


Figure 1.4: The specific solution of Example 2.1 if $u(x, 0) = 10 \sin(3\pi x/L)$. Here $\sigma = 0.5$ and $L = 3$.

(b)

What do you expect the solution to be if boundary condition [**1**] in Example 1.2 is replaced by

$$u(x, 0) = 25 \sin(\pi x/L) + 10 \sin(3\pi x/L)\,?$$

The solution is shown in Figure 1.5.

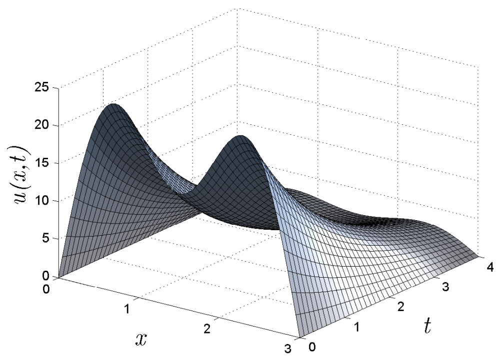


Figure 1.5: The specific solution of Example 2.1 if $u(x, 0) = 25 \sin(\pi x/L) + 10 \sin(3\pi x/L)$. Here $\sigma = 0.5$ and $L = 3$.

1.5 Foreword on Fourier analysis

In Example 1.2, after boundary conditions [**2**] and [**3**] have been imposed, (1.14) followed. Imposing boundary condition [**1**] on this solution was very straightforward, since the initial condition that was specified matched the form of (1.14), evaluated at $t = 0$, perfectly, as can be seen from (1.15). This will rarely be the case, since any arbitrary function $f(x)$ can be specified as a boundary condition. We therefore need a method to approximate an arbitrary function with functions that can easily be imposed as boundary conditions (such as e.g. $\sin(\pi x/L)$ in Example 2.1).

A series of sine and cosine functions often forms part of a general solution of a PDE. Fourier analysis is concerned with the decomposition of periodic functions into sine and cosine

Table 1.1: Three types of Fourier transformations.

Fourier Series (FS)	transforms between a periodic function and an infinite discrete series of coefficients
Fourier Transform (FT), also called the Continuous Fourier Transform (CFT)	transforms between two continuous functions defined on $(-\infty, \infty)$
Discrete Fourier Transform (DFT)	transforms between two discrete periodic series

components. When an arbitrary function is supplied as a boundary for a boundary value problem, one has to determine how this function can be represented as an infinite series of sine and/or cosine terms. Fourier analysis provides the basis for performing this task.

Fourier analysis has various generalisations, and it may be said that in general Fourier analysis is a method to transform from the physical space to the frequency space and back. There are three types of Fourier transformations available. They are listed in Table 1.1. In these notes only the Fourier series (Chapter 2) and the Fourier transform (Chapter 3) will be discussed, as only these are used in these notes in the solution of boundary value problems.

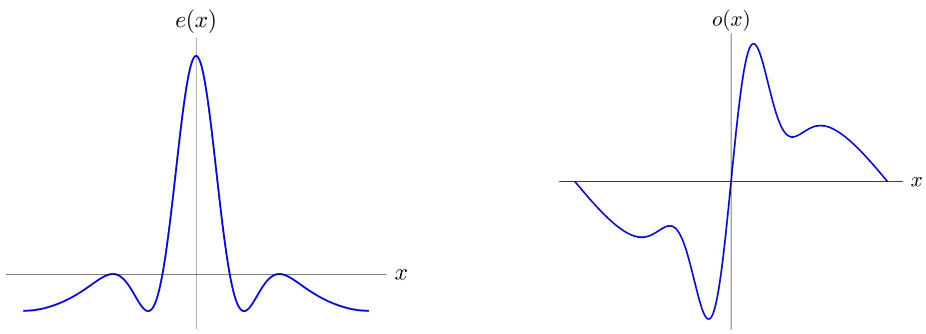
1.6 Properties of functions

Before we can continue to Fourier analysis we need to review some important properties of functions that will be utilised in the subsequent chapters.

1.6.1 Parity

"Parity has to do with 'oddness' or 'evenness' of a function. Knowing the parity of a function and using its properties, helps to reduce work when the Fourier coefficients are calculated."

A function whose graph is a mirror image about the y-axis is an *even* function. A function whose graph consists of an upside-down image on the other side of the y-axis, is an *odd* function. Figures 1.6(a) and (b) are illustrations of an even and an odd function, respectively.



(a) An even function. (b) An odd function.

Figure 1.6: An illustration of functions with parity.

DEFINITION:

A function $e(x)$ is *even* if $e(-x) = e(x)$ for all real x.

A function $o(x)$ is *odd* if $o(-x) = -o(x)$ for all real x.

Examples of even functions are x^2, x^4, e^{-x^2}, and $\cos x$. Examples of odd functions are x, x^3, $1/x$, and $\sin x$. A properly defined odd function $o(x)$ must have the property that $o(0) = 0$.

Reference to whether a function is even or odd, is collectively called the *parity* of the function. It is said that a function that is neither even nor odd has "no parity".

The term "parity" is also applied to integers. For example, if a set of functions $\{\phi_0, \phi_1, ...\}$ is given and ϕ_0, ϕ_2, ϕ_4, ... are even, while ϕ_1, ϕ_3, ... are odd, it is said that "the functions and their indices have the same parity".

Parity decomposition

Any function $f(x)$ may be decomposed into an even part and an odd part. If

$$f(x) \;=\; e(x) + o(x),$$

then the even component is

$$e(x) \;=\; \tfrac{1}{2}\left[f(x) + f(-x)\right],$$

and the odd component is

$$o(x) \;=\; \tfrac{1}{2}\left[f(x) - f(-x)\right].$$

Products

Products of functions with parity are as follows:

- even $\times$ even = even
- even $\times$ odd = odd
- odd $\times$ odd = even

CHECK BOX 1.4

Confirm that if $e(-x) = e(x)$ and $o(-x) = -o(x)$, then their product, $f(x) = e(x)o(x)$, also has the property that $f(-x) = -f(x)$, i.e. $f(x)$ is an odd function.

Do the same for the other cases.

Derivatives

All the even derivatives of a function have the same parity as the function, while the odd derivatives have opposite parity to the function.

We will illustrate this by considering the first derivative of an even function, $e(-x)$:

$$\begin{aligned} e'(-x) &= \lim_{\Delta x \to 0} \frac{e(-x+\Delta x) - e(-x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{e(x-\Delta x) - e(x)}{\Delta x} \\ &= -\lim_{\Delta x \to 0} \frac{e(x) - e(x-\Delta x)}{\Delta x} \\ &= -\lim_{\Delta x \to 0} \frac{e(x+\Delta x) - e(x)}{\Delta x} \\ &= -e'(x)\,. \end{aligned}$$

The first derivative of an even function is therefore an odd function.

CHECK BOX 1.5

Confirm that

(a) the first derivative of an odd function is an even function
(b) the second derivative of an even function is also an even function.

Integrals over a symmetric interval

Integrals over $[-K, K]$ simplify when the integrand has parity. Let $e(x)$ be even and $o(x)$ be odd, then

$$\int_{-K}^{K} e(x)\,dx = 2\int_{0}^{K} e(x)\,dx\,,$$

and

$$\int_{-K}^{K} o(x)\,dx = 0\,.$$

1.6.2 Periodicity

"Since Fourier analysis has to do with periodic functions, we shall give the definition of a periodic function and list some properties thereof."

Periodic functions such as $\sin x$ and $\cos x$ are well known. These functions repeat the same pattern on consecutive intervals with width 2π. The interval width for repetition is called the *period*.

DEFINITION:

A function $p(x)$ is *periodic* with *period* K if

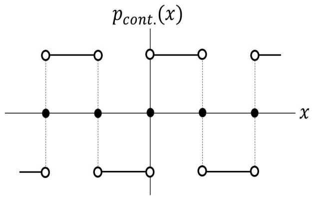
$$p(x+K) = p(x) \qquad \text{for all real } x.$$

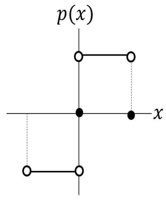
Table 1.2: Examples of well known periodic functions.

Square wave function

$$p_{cont.}(x) = \begin{cases} -1, & x \in (2k-1, 2k) \\ 0, & x = 2k \\ 1, & x \in (2k, 2k+1) \\ 0, & x = 2k+1 \end{cases}$$

for $k \in \mathbb{Z}$

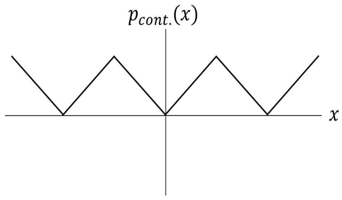


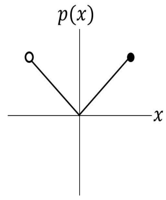


Roof-top function

$$p_{cont.}(x) = \begin{cases} -x+2k, & x \in (2k-1, 2k) \\ x-2k, & x \in [2k, 2k+1] \end{cases}$$

for $k \in \mathbb{Z}$

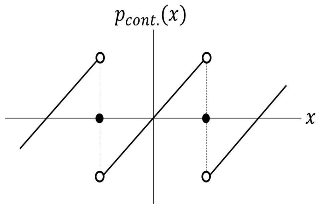


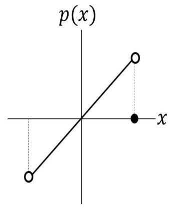


Saw-tooth function

$$p_{cont.}(x) = \begin{cases} x-2k, & x \in (2k-1, 2k+1) \\ 0, & x = 2k+1 \end{cases}$$

for $k \in \mathbb{Z}$





The terminology "$p(x)$ is K-periodic" is often also employed. Note that if a function is K-periodic, then it is automatically also $2K$-periodic, $3K$-periodic, etc. The period is, however, defined as the smallest interval width over which the function is periodic.

If the periodic function is known over one period, then it is known for all real x, and therefore a periodic function is often given by simply supplying its value over only one period or *window*. For example, the square wave (see Table 1.2) may be given as

$$p(x) = \begin{cases} -1, & x \in (-1, 0), \\ 0, & x = 0, \\ 1, & x \in (0, 1), \\ 0, & x = 1. \end{cases}$$

The periodic function for the square wave is called the *periodic continuation* of $p(x)$, and we shall denote it by $p_{cont.}(x)$. Three examples of periodic functions that occur often, are listed in Table 1.2. We denote the set of integers by $\mathbb{Z}$, i.e.

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}.$$

The periodic continuations and window periods of these periodic functions are also illustrated.

1.6.3 Function norms and the inner product

"As will be shown in Paragraph 2.3, the Fourier approximation formulae are derived from the fact that the sine and cosine functions in the series form a complete orthogonal set.

In order to understand the orthogonality concept, the idea of the 'magnitude of a function' (the function norm) and 'how close two functions are to each other' (the inner product), must first be introduced."

It may be helpful to utilise the analogy between vector norms and function norms, and likewise the analogy between vector dot products and function inner products, in order to see where the definitions of function norms and function inner products come from. In a more general setting, these concepts are actually considered entirely equivalent.

Let us recall the idea of vector dot products and vector norms.

Let $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$, then the *dot product* of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a}^T\mathbf{b} \ = \ \sum_{j=1}^{n} \overline{a}_j b_j,$$

and the (Euclidean) *norm* of $\mathbf{a}$ is

$$\|\mathbf{a}\| \ = \ \sqrt{\mathbf{a}^T\mathbf{a}} \ = \ \sqrt{\sum_{j=1}^{n} \overline{a_j} a_j}\,,$$

where the overbar denotes the complex conjugate.

For functions $f(x)$ and $g(x) : [a, b] \to \mathbb{C}$ (i.e. the function values may be complex but x is real on $[a, b]$), we define:

DEFINITION: The *inner product* of $f(x)$ and $g(x)$ on $[a, b]$ is

$$\langle f, g \rangle \ = \ \int_a^b \overline{f(x)} g(x)\, dx \tag{1.17}$$

and

DEFINITION: The *norm* of $f(x)$ on the interval $[a, b]$ is

$$\|f(x)\| \ = \ \sqrt{\langle f, f \rangle} \ = \ \sqrt{\int_a^b \overline{f(x)} f(x)\, dx} \tag{1.18}$$

where, again, the overbar denotes the complex conjugate.

Note how summation in the vector sense is replaced by integration in the function sense.

A function is *normalised* when it is scaled such that its norm is one.

EXAMPLE 1.4

Normalise $f(x) = x^2$ on the interval $[-1, 1]$.

$$\begin{aligned} f(x)_{normalised} &= \frac{f(x)}{\|f(x)\|} \\ &= \frac{x^2}{\sqrt{\int_{-1}^{1} x^4 dx}} \\ &= \sqrt{\frac{5}{2}} x^2 \end{aligned}$$

The physical interpretation of the dot and inner products may be useful: Two normalised vectors point in approximately the same direction when their dot product is close to one. Likewise, two normalised functions are "close to each other" when their inner product is close to one.

When the dot product of two vectors is zero, they make a 90° angle with respect to each other, and the projection of one vector on the other is zero. One could say that when the dot product of two vectors is zero, then the one vector "contains nothing" of the other vector. Similarly, two functions "contain nothing of each other" if their inner product is zero. An example is $\langle \sin(x), x^2 \rangle = 0$ on any symmetric interval. The Taylor series of $\sin x$, i.e. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$, illustrates this – there is no x^2 in the series.

1.6.3.1 Orthogonal functions

Two functions are orthogonal over a given interval if their inner product over that interval is zero.

DEFINITION: $f(x)$ and $g(x)$ is *orthogonal* on $[a, b]$ if

$$\int_a^b \overline{f(x)} g(x)\, dx = 0\,. \tag{1.19}$$

CHECK BOX 1.6

(a) Show that $f(x) = x - \frac{3}{4}$ and $g(x) = x^2$ are orthogonal on $[0, 1]$.

(b) Show that $f(x) = \sin x$ and $g(x) = \cos x$ are orthogonal on $[-\pi, \pi]$.

1.6.3.2 Orthonormal functions

A set of orthonormal vectors form a convenient basis for a vector space and it simplifies the obtaining of coefficients for the vector representation in this basis. Likewise a set of orthonormal functions on $[a, b]$, will simplify the finding of coefficients of a function representation in the basis spanned by the set of functions.

DEFINITION: The set of functions $\{\phi_j(x),\ j = 0, 1, ..., N\}$ are *orthonormal* on $[a, b]$ if

$$\int_a^b \overline{\phi_j(x)}\phi_k(x)\,dx \;=\; \begin{cases} 0, & \text{when} \quad j \neq k, \\ 1, & \text{when} \quad j = k. \end{cases} \tag{1.20}$$

CHECK BOX 1.7

Show that the following set of functions is an orthonormal set on the interval $[-1, 2]$:

$$\phi_0(x) \;=\; \frac{1}{\sqrt{3}}\,; \qquad \phi_1(x) \;=\; \frac{1}{3}\,(2x-1)\,; \qquad \phi_2(x) \;=\; \sqrt{\frac{20}{27}}\left(x^2 - x - \frac{1}{2}\right).$$

1.7 Approximations in general

"Approximating one function by another does not necessarily need the concept of orthogonality. In this section we shall discuss approximation techniques in general without referring to orthogonality."

We first consider the general problem of approximating a given function $f(x)$ on a given interval $[a, b]$ by another function $p(x)$ that contains a number of parameters. These parameters must be chosen in such a way that the approximation is "as good as possible" in some sense.

Suppose $p(x)$ is a linear combination of a set of basis functions, while $f(x)$ may or may not lie in the function space spanned by these basis functions. We shall now discuss the so called *least squares approximation* idea.

1.7.1 The least squares approximation

The least squares approximation finds $p(x)$ such that the function norm of the difference between $f(x)$ and $p(x)$ over the given interval is as small as possible.

Let

$$E \;=\; \|f(x) - p(x)\|^2 \;=\; \int_a^b [f(x) - p(x)]^2\,dx\,.$$

E is then minimised with respect to every parameter in $p(x)$.

EXAMPLE 1.5

Approximate $f(x) = e^x$ by the parabola

$$p(x) = ax^2 + bx + c\,,$$

on the interval $[0, 1]$.

Let

$$E = \int_0^1 [f(x) - p(x)]^2 \, dx\,.$$

E is a minimum with respect to a, b and c if

$$\frac{\partial E}{\partial c} = 0, \quad \frac{\partial E}{\partial b} = 0, \quad \frac{\partial E}{\partial a} = 0.$$

This leads to the following set of equations in a, b and c:

$$\begin{aligned}
a\int_0^1 x^2 dx &+ b\int_0^1 x dx &+ c\int_0^1 1 dx &= \int_0^1 e^x dx,\\
a\int_0^1 x^3 dx &+ b\int_0^1 x^2 dx &+ c\int_0^1 x dx &= \int_0^1 x e^x dx,\\
a\int_0^1 x^4 dx &+ b\int_0^1 x^3 dx &+ c\int_0^1 x^2 dx &= \int_0^1 x^2 e^x dx,
\end{aligned}$$

or

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} e-1 \\ 1 \\ e-2 \end{bmatrix} \tag{1.21}$$

with solution $a = 0.8392$, $b = 0.8511$ and $c = 1.0130$.

Figure 1.7 shows both $f(x) = e^x$ and $p(x) = ax^2 + bx + c$ on the same axes. Note how well $p(x)$ approximates $f(x)$, on the interval $[0, 1]$ and how the approximation quickly deteriorates outside the interval.

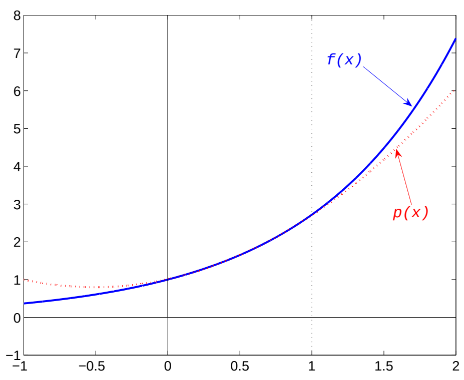


Figure 1.7: The function, $f(x)$, and its approximation, $p(x)$.

1.7.2 The least squares approximation with orthogonal functions

"In Example 1.5 (1.21), the matrix on the left was full and the full system of three equations in three unknowns had to be solved. Is there a way to simplify the system, for example, can the matrix be made diagonal?

The answer is, yes: If $p(x)$ is expressed as a combination of orthogonal functions, then the matrix is diagonal, and the system is easy to solve."

Although orthogonality is defined for complex functions, we shall simplify the ideas in this section by considering only real functions.

Let $\{\phi_0(x), \phi_1(x), ..., \phi_N(x)\}$ be an orthogonal set of functions on the interval $[a, b]$:

$$\int_a^b \phi_j(x)\phi_k(x)\,dx \;=\; \begin{cases} 0 & \text{when} \quad j \neq k\,, \\ \|\phi_k\|^2 & \text{when} \quad j = k\,. \end{cases} \tag{1.22}$$

Let the approximating function be,

$$p(x) \;=\; \sum_{j=0}^{N} c_j\phi_j(x)\,,$$

and let

$$E \;=\; \int_a^b [f(x) - p(x)]^2\,dx\,.$$

We shall now minimise E with respect to every c_k, $k = 0, 1, ..., N$:

$$\begin{aligned}
\frac{\partial E}{\partial c_k} &= \frac{\partial}{\partial c_k}\int_a^b \Big[f(x) - c_0\phi_0(x) - c_1\phi_1(x) - ... - c_k\phi_k(x) - ... - c_N\phi_N(x)\Big]^2 dx \\
&= 2\int_a^b \Big[f(x) - c_0\phi_0(x) - c_1\phi_1(x) - ... - c_N\phi_N(x)\Big] \times \Big(-\phi_k(x)\Big)\,dx \\
&= -2\int_a^b f(x)\phi_k(x)\,dx + 2c_0\int_a^b \phi_0(x)\phi_k(x)\,dx + ... + 2c_k\int_a^b \phi_k(x)\phi_k(x)\,dx + ... \\
&= -2\int_a^b f(x)\phi_k(x)\,dx + 0 + 0 + ... + 2c_k\|\phi_k\|^2 + ... \\
&= 0\,,
\end{aligned}$$

therefore

$$c_k \;=\; \frac{1}{\|\phi_k\|^2}\int_a^b f(x)\phi_k(x)\,dx. \tag{1.23}$$

This expression was derived by minimising by means of derivatives. A different approach follows from the "geometrical" observation that $\|f(x) - p(x)\|$ is a minimum if the inner product of $f(x) - p(x)$ with every $\phi_j(x)$ is zero. This approach gives the same result as (1.23).

EXAMPLE 1.6

Consider the following orthogonal functions on $[-1, 1]$:

$$\begin{aligned}
\phi_0(x) &= 1\,, & \text{where} \quad \|\phi_0\|^2 &= 2\,,\\
\phi_1(x) &= x\,, & \text{where} \quad \|\phi_1\|^2 &= 2/3\,,\\
\phi_2(x) &= x^2 - \tfrac{1}{3}\,, & \text{where} \quad \|\phi_2\|^2 &= 8/45\,.
\end{aligned}$$

Find the least squares approximation of the form

$$p(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x)$$

for $f(x) = e^x$ on the interval $[-1, 1]$.

From (1.23) it follows that

$$c_0 = \frac{1}{\|\phi_0\|^2}\int_{-1}^{1} f(x)\phi_0(x)\,dx = \frac{1}{2}\int_{-1}^{1} e^x\,dx = \frac{e - e^{-1}}{2} = 1.1752$$

$$c_1 = \frac{1}{\|\phi_1\|^2}\int_{-1}^{1} f(x)\phi_1(x)\,dx = \frac{3}{2}\int_{-1}^{1} xe^x\,dx = 3e^{-1} = 1.1036$$

$$c_2 = \frac{1}{\|\phi_2\|^2}\int_{-1}^{1} f(x)\phi_2(x)\,dx = \frac{45}{8}\int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)e^x\,dx = \frac{15e - 105e^{-1}}{4} = 0.5367$$

The least squares approximation is

$$p(x) = 1.1752 + 1.1036x + 0.5367\left(x^2 - \frac{1}{3}\right) = 0.5367x^2 + 1.1036x + 0.9963\,.$$

Figure 1.8 shows graphs of $f(x) = e^x$ as well as $p(x)$ on the same axes. The parabola approximates the function well on the interval, but the approximation deteriorates outside the interval.

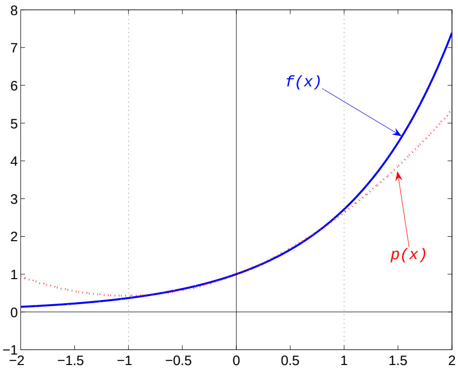


Figure 1.8: The function, $f(x)$, and its approximation, $p(x)$.

CHECK BOX 1.8

Check that the following set of functions are orthogonal on the interval $[-\pi, \pi]$:

$$\phi_0(x) = 1\,, \quad \phi_1(x) = \cos(x)\,, \quad \phi_2(x) = \cos(2x)\,.$$

Using this set of functions:

(a) Find the least squares approximation of $f(x) = x^2$ on $[-\pi, \pi]$.

Answer: $c_0 = \pi^2/3, \quad c_1 = -4, \quad c_2 = 1.$

Figure 1.9 shows graphs of $p(x)$ and $f(x)$ on the same axes.

(b) Take the same set of orthogonal functions and find the least squares solution of $f(x) = \sin^2 x$ on $[-\pi, \pi]$. Draw graphs of the function and its approximation. Can you explain the result?

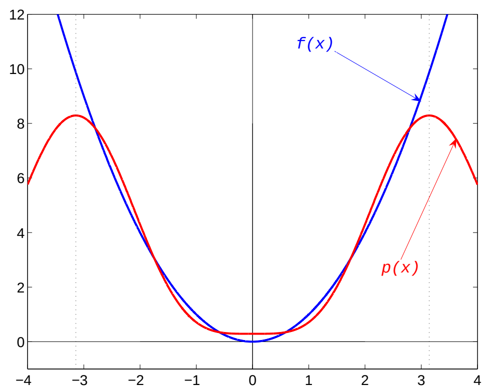


Figure 1.9: The function, $f(x)$, and its approximation, $p(x)$.

PROBLEM SET 1: INTRODUCTION

1. Solve

$$ku_t = \sigma u_{xx} \qquad \text{on } \; x \in [0,1], \quad t \in [0,\infty)\,,$$

where

(a) $k = 1$ and $\sigma = 4$, with

$$u(x,0) = \sin(3\pi x) + x\,, \quad u(0,t) = 0\,, \quad \text{and} \quad u(1,t) = 1\,.$$

(b) $k = 1$ and $\sigma = 0.16$, with

$$u(x,0) = \sin(3\pi x) + \tfrac{1}{4}x\,, \quad u(0,t) = 0\,, \quad \text{and} \quad u(1,t) = \tfrac{1}{4}\,.$$

(c) $k = 1$ and $\sigma = 0.09$, with

$$u(x,0) = 2\sin(5\pi x) + \tfrac{1}{2}x\,, \quad u(0,t) = 0\,, \quad \text{and} \quad u(1,t) = \tfrac{1}{2}\,.$$

(d) $k = 5$ and $\sigma = 1$, with

$$u(x,0) = 60\cos(3.5\pi x)\,, \quad u_x(0,t) = 0\,, \quad \text{and} \quad u(1,t) = 0\,.$$

(e) $k = 1$ and $\sigma = 4$, with

$$u(x,0) = 20\cos(2.5\pi x)\,, \quad u_x(0,t) = 0\,, \quad \text{and} \quad u(1,t) = 0\,.$$

Assume (without derivation) that the general solution is of the form

$$u(x,t) = e^{-\frac{\sigma\lambda^2 t}{k}}\left[A\cos(\lambda x) + B\sin(\lambda x)\right] + Cx + D\,.$$

Plot the solutions.

2. Assume that A, B, L, α, β, σ, and γ are all positive constants.

For each of the following three problems, find the expression for the steady-state of u, that is, the eventual shape that $u(x,t)$ will assume when $t \to \infty$. No initial condition is given here, since it has no effect on the steady state solution. Also plot these steady-state solutions (they are 2D graphs).
Hint: The "steady-state" simply means that nothing is changing anymore, i.e. $u_t = 0$.

(a)

$$u_t = \sigma u_{xx} + \beta \qquad \text{on } \; x \in [0,1], \quad t \in [0,\infty)\,,$$

with

$$u(0,t) = 0\,, \quad \text{and} \quad u(1,t) = A\,.$$

(b)

$$u_t = \sigma u_{xx} + \gamma u \qquad \text{on } \; x \in [0,L], \quad t \in [0,\infty)\,,$$

with

$$u_x(0,t) = 0\,, \quad \text{and} \quad u(L,t) = B\,.$$

(c)

$$u_t = \sigma u_{xx} + \alpha u \qquad \text{on } x \in [0,L], \quad t \in [0,\infty)\,,$$

with

$$u_x(0,t) = A\,, \quad \text{and} \quad u(L,t) = 0\,.$$

3. **(a)** If $p_m(x)$ and $p_n(x)$ are complex functions, then orthogonality requires that

$$\int_{-L}^{L} \overline{p_m(x)} p_n(x) dx = 0$$

where the overbar denotes the complex conjugate of the function. Show that the following set of functions

$$e^{ik\pi x/L}$$

is orthogonal on $[-L,L]$ for $k \in \mathbb{Z}$.

(b) Derive the formula for solving coefficients, c_k, if a function $f(x)$ is expanded as

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\pi kx/L}\,. \tag{1}$$

Do it as follows: Multiply (1) on both sides by $e^{-i\pi nx/L}$ and then integrate with respect to x over $[-L,L]$. Use the result from **3(a)** and simplify.

Chapter 2

The Fourier series

2.1 Introduction

"This section shows how a Fourier series is obtained, without derivation of the formulae. It also illustrates how a truncated Fourier series approximates the function from which it was calculated. The derivation of the formulae for the Fourier coefficients will be done in Paragraph 2.3."

A Fourier series is an infinite series of sine and cosine functions that approximates a given function $f(x)$ over the interval $[-L, L]$. A *truncated* Fourier series is a finite series consisting of the first N terms of the infinite Fourier series. Such a truncated series is often called a *Fourier approximation.* Denote the truncated Fourier series of $f(x)$ with N terms, by $f_N(x)$ and the infinite Fourier series by $f_\infty(x)$.

We now state without derivation that

$$\boxed{f_N(x) \;=\; \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(k\pi x/L) + \sum_{k=1}^{N} b_k \sin(k\pi x/L)\,,} \tag{2.1}$$

where

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\,dx \\
a_k &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(k\pi x/L)\,dx \qquad k = 1, 2, ..., N \\
b_k &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(k\pi x/L)\,dx \qquad k = 1, 2, ..., N
\end{aligned} \tag{2.2}$$

The constants $a_k,\ k = 0, 1, ...$ and $b_k,\ k = 1, 2, ...$, are called the *Fourier coefficients.*

The *basis functions* used in the expansion are $\sin(k\pi x/L)$ and $\cos(k\pi x/L)$. In this case the basis functions form an *orthogonal set* as was discussed in Paragraph 1.7.2.

The index k is often called the *frequency*, because for larger k, more oscillations per interval are witnessed in the relevant basis function. The factor $(k\pi/L)$ corresponds to the physical concept of *angular frequency* of an oscillation.

The value of each coefficient is an indication of "how much" of the basis function associated with it, is "present" in the function $f(x)$. For example, if $f(x)$ is an even function, then only even basis functions (i.e. the cosines) will be present in the function and all the b_k's will be zero. If $f(x)$ displays wild oscillations then those Fourier coefficients with larger k will be relatively greater than for a function that does not show oscillations.

EXAMPLE 2.1

Find the Fourier series of

$$f(x) \;=\; \begin{cases} -1, & x \in (-\pi, 0), \\ 1, & x \in (0, \pi). \end{cases}$$

Since $f(x)$ is odd, $a_k = 0$, for all $k = 0, 1, ...$ The b_k coefficients are obtained as follows:

$$b_k \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\, dx\,.$$

This Fourier coefficient may be calculated:

$$\begin{aligned} b_k \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\, dx \;&=\; \frac{1}{\pi}\int_{-\pi}^{0} -\sin(kx)\, dx + \frac{1}{\pi}\int_{0}^{\pi} \sin(kx)\, dx \\ &=\; \frac{1}{k\pi}\Big[\cos(kx)\Big]_{-\pi}^{0} + \frac{1}{k\pi}\Big[-\cos(kx)\Big]_{0}^{\pi} \\ &=\; \frac{1}{k\pi}\Big[1-(-1)^k\Big] + \frac{1}{k\pi}\Big[-(-1)^k-(-1)\Big] \\ &=\; \begin{cases} \dfrac{4}{k\pi} & \text{for } k \text{ odd,} \\ 0 & \text{for } k \text{ even.} \end{cases} \end{aligned}$$

The Fourier series is therefore given by

$$f_\infty(x) \;=\; \frac{4}{\pi}\left[\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + ...\right].$$

Odd numbers may be expressed as $(2j-1)$ for $j = 1, 2, ...$ and therefore the truncated Fourier series may be expressed as follows,

$$f_N(x) \;=\; \frac{4}{\pi}\sum_{j=1}^{[(N+1)/2]} \frac{\sin((2j-1)x)}{2j-1}, \qquad x \in [-L, L]\,.$$

Actually $f_N(x)$, for $x \in \mathbb{R}$, is an approximation of the periodic continuation of $f(x)$, i.e. of the following function,

$$f_{cont.}(x) = \begin{cases} -1\,, & x \in ((2k-1)\pi, 2k\pi), \\ 1\,, & x \in (2k\pi, (2k+1)\pi), \end{cases} \qquad k \in \mathbb{Z}, \quad x \in \mathbb{R}.$$

Figure 2.1 shows $f_{continued}(x)$ as well as $f_N(x)$ for $N = 1, 3, 5, 11$, and 23. Notice how the approximation improves with increasing N. Also notice how the Fourier approximation finds it difficult to approximate well near the discontinuities in $f_{continued}(x)$. This behaviour (overshoot and strong oscillations close to the discontinuity) is called the *Gibbs phenomenon*. If the periodic continuation of the function $f(x)$ does not contain any discontinuities, then the Gibbs phenomenon is absent. Example 2.2 illustrates this.

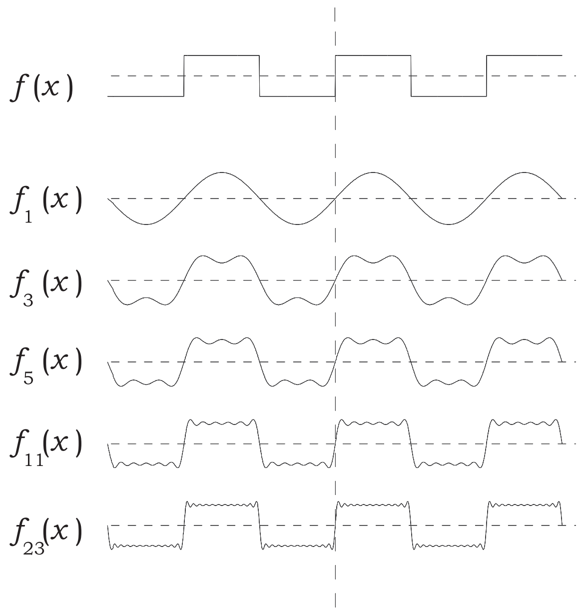


Figure 2.1: The Fourier series of the square wave.

EXAMPLE 2.2

Find the Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$.

Since $f(x)$ is even, all $b_k = 0$.

The relevant Fourier coefficients are

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{1}{\pi}\left[\frac{\pi^2}{2}\right] = \pi$$

and

$$a_k = \frac{2}{\pi}\int_0^\pi x\cos(kx)dx = \frac{2}{k^2\pi}[(-1)^k - 1] = \begin{cases} \dfrac{-4}{k^2\pi} & \text{for } k \text{ odd}, \\ 0 & \text{for } k \text{ even}. \end{cases}$$

Therefore, the Fourier series approximating $f(x)$ is

$$f_\infty(x) \;=\; \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + ...\right],$$

or, in terms of the summation symbol, the truncated Fourier series may be expressed as

$$f_N(x) \;=\; \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=1}^{[(N+1)/2]}\left(\frac{\cos((2j-1)x)}{(2j-1)^2}\right).$$

Figure 2.2 shows a periodic continuation of $f(x)$ as well as $f_N(x)$ for $N = 1, 3, 5, 7$, and 15. Notice that the Gibbs phenomenon is absent.

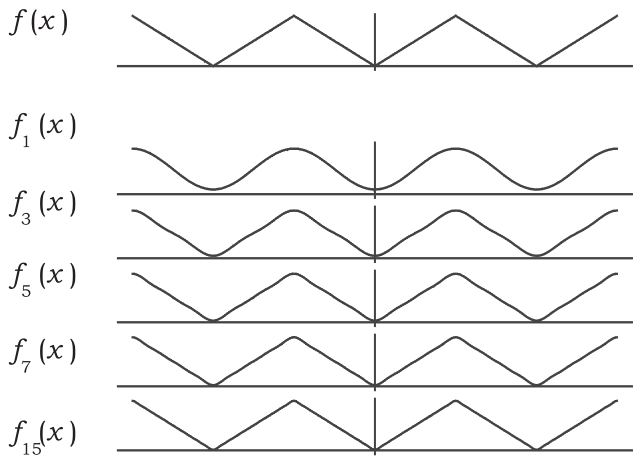


Figure 2.2: The Fourier series of the "roof-top" function.

Convergence

We shall not discuss the convergence of Fourier series in detail, but only mention the following:

If $f(x)$ is continuous at x_0, then

$$\lim_{N\to\infty} f_N(x_0) \;=\; f(x_0)\,. \tag{2.3}$$

If $f(x)$ is discontinuous at x_0, then

$$\lim_{N\to\infty} f_N(x) \;=\; \frac{1}{2}\Big[f(x_0-0) + f(x_0+0)\Big]\,. \tag{2.4}$$

Here the notation $f(x_0 - 0)$ is an abbreviation for $\lim_{\epsilon\to 0} f(x_0 - \epsilon)$ and similarly, $f(x_0 + 0)$ is an abbreviation for $\lim_{\epsilon\to 0} f(x_0 + \epsilon)$.

For example, the Fourier series of Example 2.1 is discontinuous at $x_0 = 0$, and therefore the average over the discontinuity is $\frac{1}{2}(1-1) = 0$. This may be confirmed by calculating $f_\infty(0)$:

$$f_\infty(0) \;=\; \frac{4}{\pi}\left[\sin(0) + \frac{\sin(0)}{3} + \frac{\sin(0)}{5} + ...\right] \;=\; 0\,.$$

2.2 Orthogonality of the Fourier functions

"In this section we show that the sine and cosine functions used in the Fourier series form an orthogonal set.

Before we start, you are reminded of the following identities:

$$
\begin{aligned}
\sin k\pi &= 0 \quad \text{for} \quad k \in \mathbb{Z}\,, \\
\cos k\pi &= (-1)^k \quad \text{for} \quad k \in \mathbb{Z}\,, \\
\cos A \cos B &= \tfrac{1}{2}[\cos(A-B) + \cos(A+B)]\,, \\
\sin A \sin B &= \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]\,, \\
\sin A \cos B &= \tfrac{1}{2}[\sin(A-B) + \sin(A+B)]\,.
\end{aligned}
$$

These identities will be needed subsequently."

The Fourier basis functions

Consider the following set of functions, $\{\phi_k(x)\}$, $k = 0, 1, ..., N$ with

$$
\begin{aligned}
\phi_0(x) &= 1 \\
\phi_{2k}(x) &= \cos(k\pi x/L) \quad \text{for} \quad k = 1, 2, ..., N \\
\phi_{2k-1}(x) &= \sin(k\pi x/L) \quad \text{for} \quad k = 1, 2, ..., N\,.
\end{aligned}
$$

We shall show that this set is orthogonal on the interval $[-L, L]$.

We firstly check that the even functions (i.e. the cosines and 1) are orthogonal:

$$
\begin{aligned}
&\int_{-L}^{L} \phi_{2j}(x)\phi_{2k}(x)\;dx \\
&= \int_{-L}^{L} \cos(j\pi x/L)\cos(k\pi x/L)\;dx \\
&= \begin{cases}
\displaystyle\int_{-L}^{L} \tfrac{1}{2}(\cos(j\pi x/L - k\pi x/L) + \cos(j\pi x/L + k\pi x/L))\;dx & \text{if} \quad j \neq k \\
\displaystyle\int_{-L}^{L} \tfrac{1}{2}(1 + \cos(2k\pi x/L))\;dx & \text{if} \quad j = k \neq 0 \\
\displaystyle\int_{-L}^{L} 1\;dx & \text{if} \quad j = k = 0
\end{cases} \\
&= \begin{cases}
\dfrac{1}{2}\left[\dfrac{\sin((j-k)\pi x/L)}{(j-k)\pi/L} + \dfrac{\sin((j+k)\pi x/L)}{(j+k)\pi/L}\right]_{-L}^{L} & \text{if} \quad j \neq k \\
\dfrac{1}{2}\left[x + \dfrac{\sin(2\pi k x/L)}{2\pi k/L}\right]_{-L}^{L} & \text{if} \quad j = k \neq 1 \\
\Big[x\Big]_{-L}^{L} & \text{if} \quad j = k = 0
\end{cases}
\end{aligned}
$$

yielding

$$\int_{-L}^{L} \phi_{2j}(x)\phi_{2k}(x)\,dx = \begin{cases} 0 & \text{if} \quad j \neq k \\ L & \text{if} \quad j = k \neq 0 \\ 2L & \text{if} \quad j = k = 0\,. \end{cases}$$

Next we check that the odd functions (i.e. the sines) are orthogonal:

$$\begin{aligned} &\int_{-L}^{L} \phi_{2j-1}(x)\phi_{2k-1}(x)\,dx \\ &= \int_{-L}^{L} \sin(j\pi x/L)\sin(k\pi x/L)\,dx \\ &= \begin{cases} \displaystyle\int_{-L}^{L} \tfrac{1}{2}(\cos(j\pi x/L - k\pi x/L) - \cos(j\pi x/L + k\pi x/L))\,dx & \text{if} \quad j \neq k \\ \displaystyle\int_{-L}^{L} \tfrac{1}{2}(1 - \cos(2k\pi x/L))\,dx & \text{if} \quad j = k \end{cases} \\ &= \begin{cases} \displaystyle\frac{1}{2}\left[\frac{\sin((j-k)\pi x/L)}{(j-k)\pi/L} - \frac{\sin((j+k)\pi x/L)}{(j+k)\pi/L}\right]_{-L}^{L} & \text{if} \quad j \neq k \\ \displaystyle\frac{1}{2}\left[x - \frac{\sin(2k\pi x/L)}{2k\pi/L}\right]_{-L}^{L} & \text{if} \quad j = k \end{cases} \\ &= \begin{cases} 0 & \text{if} \quad j \neq k \\ L & \text{if} \quad j = k\,. \end{cases} \end{aligned}$$

Two functions with opposite parity (i.e. one is odd, the other is even) are obviously orthogonal on $[-L, L]$, so that we do not need to check this by integrating out.

To summarise, this set of functions has the following property,

$$\int_{-L}^{L} \phi_j(x)\phi_k(x)\,dx = \begin{cases} 0 & j \neq k \\ L & j = k \neq 0 \\ 2L & j = k = 0\,. \end{cases}$$

Figure 2.3 shows the first seven Fourier functions.

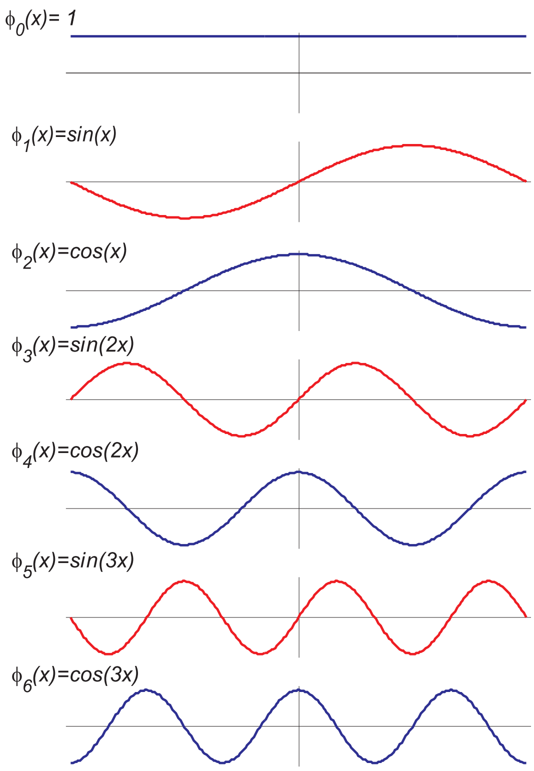


Figure 2.3: The first seven Fourier functions.

2.3 Derivation of the Fourier series

"Two concepts have been established now:

(1) The Fourier functions form an orthogonal set,

(2) Using orthogonal functions to approximate a given function over the interval of orthogonality, creates a diagonal system that is easy to solve.

Let us now put these two ideas together."

You are reminded that the set of Fourier functions are given by

$$\begin{aligned}\phi_0(x) &= 1 \\ \phi_{2k}(x) &= \cos(k\pi x/L) \quad \text{for} \quad k = 1, 2, ..., N \\ \phi_{2k-1}(x) &= \sin(k\pi x/L) \quad \text{for} \quad k = 1, 2, ..., N\,.\end{aligned}$$

For a given fixed N, we consider the space of all functions of the form

$$f_N(x) = \sum_{j=0}^{2N} c_j \phi_j(x)\,. \tag{2.5}$$

$f_N(x)$ is sometimes called a *trigonometric polynomial of degree* N.

We may want to approximate a given function $f(x) : [-L, L] \to \mathbb{R}$, by $f_N(x)$.

If $f_N(x)$ is the least squares approximation to $f(x)$ on $[-L, L]$, the theory derived in Paragraph 1.7.2 shows that the coefficients of the least squares approximation are found from

$$c_j = \frac{1}{L}\int_{-L}^{L} f(x)\phi_j(x)\, dx \qquad \text{for} \quad j = 1, ..., 2N \tag{2.6}$$

and

$$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx\,. \tag{2.7}$$

Unfortunately the fact that $\|\phi_0\|^2 = 2L$ instead of L like all the other basis functions, forces us to write it down separately.

The variable x in (2.6) is simply an integration variable, and should not be confused with the x in (2.5) where it denotes the independent variable on which the function is defined. Therefore, in (2.6), x may be replaced by any other symbol.

Relabeling the coefficients

A more convenient way to express the series in (2.5), is to relabel the coefficients so that the index is the same as the frequency, and so that each type of function, sine and cosine, has its own coefficient.

Let

$$\begin{aligned} a_0 &= 2c_0 \\ a_k &= c_{2k} && \text{for} \quad k = 1, 2, ..., N \\ b_k &= c_{2k-1} && \text{for} \quad k = 1, 2, ..., N. \end{aligned}$$

The *Fourier series*, $f_N(x)$, and the *Fourier coefficients*, $\{a_k,\ k = 0, 1, ...\}$ and $\{b_k,\ k = 1, 2, ...\}$, may now be expressed as follows:

Fourier Series

$$\boxed{f_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(k\pi x/L) + \sum_{k=1}^{N} b_k \sin(k\pi x/L)} \tag{2.8}$$

Fourier Series Coefficients

$$\boxed{\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\, dx \\ a_k &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(k\pi x/L)\, dx && \text{for} \quad k = 1, 2, ..., N \\ b_k &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(k\pi x/L)\, dx && \text{for} \quad k = 1, 2, ..., N \end{aligned}} \tag{2.9}$$

The Fourier series representation of $f(x)$ is therefore

$$\begin{aligned} f_N(x) = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx &+ \sum_{k=1}^{N}\left[\frac{1}{L}\int_{-L}^{L} f(x)\cos(k\pi x/L)\, dx\right]\cos(k\pi x/L) \\ &+ \sum_{k=1}^{N}\left[\frac{1}{L}\int_{-L}^{L} f(x)\sin(k\pi x/L)\, dx\right]\sin(k\pi x/L)\,, \end{aligned} \tag{2.10}$$

where $f_N(x)$ approximates $f(x)$ on the interval $[-L, L]$. However, note that $f_N(x)$ is actually a periodic function with period $2L$. On the real axis, $x \in (-\infty, \infty)$, $f_N(x)$ approximates the *periodic continuation* of $f(x)$.

When N is finite, we refer to (2.8) as a *Fourier approximation* or *truncated Fourier series* of $f(x)$. If $N \to \infty$, then (2.8) is called a *Fourier series.*

2.4 Derivation of the complex Fourier series

"In this section we show that, by using complex variables, a more symmetric and compact formula may be obtained for the Fourier series. Before we start, you are reminded of the following:

$$\begin{aligned} e^{\pm i\theta} &= \cos\theta \pm i\sin\theta\,, \\ \sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i}\,, \\ \cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2}\,. \end{aligned}$$

These identities will be used throughout this section."

Let $\{a_k\}$ and $\{b_k\}$ be the Fourier coefficients of the function $f(x)$ defined on $[-L, L]$. We introduce a new set of coefficients, viz. $\{c_k,\ k = -N, -N+1, ..., -1, 0, 1, ..., N-1, N\}$. These are given by

$$\begin{aligned} c_0 &= \tfrac{1}{2}a_0 \\ \left.\begin{aligned} c_k &= \tfrac{1}{2}(a_k - ib_k) \\ c_{-k} &= \tfrac{1}{2}(a_k + ib_k) \end{aligned}\right\} &\quad \text{for} \quad k = 1, 2, ..., N\,. \end{aligned} \tag{2.11}$$

The original coefficients may be found from $\{c_k\}$ by means of the following formulae

$$\begin{aligned} a_0 &= 2c_0 \\ \left.\begin{aligned} a_k &= c_k + c_{-k} \\ b_k &= i(c_k - c_{-k}) \end{aligned}\right\} &\quad \text{for} \quad k = 1, 2, ..., N\,. \end{aligned} \tag{2.12}$$

The Fourier series may then be expressed as

$$\begin{aligned} f_N(x) &= \frac{a_0}{2} + \sum_{k=1}^{N} \left[a_k \cos(k\pi x/L) + b_k \sin(k\pi x/L)\right] \\ &= c_0 + \sum_{k=1}^{N} \left[(c_k + c_{-k})\cos(k\pi x/L) + i(c_k - c_{-k})\sin(k\pi x/L)\right] \\ &= c_0 + \sum_{k=1}^{N} \left[c_k\left(\cos(k\pi x/L) + i\sin(k\pi x/L)\right) + c_{-k}\left(\cos(k\pi x/L) - i\sin(k\pi x/L)\right)\right] \\ &= c_0 e^0 + \sum_{k=1}^{N} \left[c_k e^{ik\pi x/L} + c_{-k} e^{-ik\pi x/L}\right]\,. \end{aligned}$$

The complex Fourier series may therefore be written as:

Complex Fourier Series

$$\boxed{f_N(x) = \sum_{k=-N}^{N} c_k e^{ik\pi x/L}} \tag{2.13}$$

A new formula for the Fourier coefficients can be obtained as follows:

$$\begin{aligned}
c_{\pm k} &= \tfrac{1}{2}(a_k \mp i b_k) \\
&= \frac{1}{2}\left[\frac{1}{L}\int_{-L}^{L} f(x)\cos(k\pi x/L)\,dx \mp \frac{i}{L}\int_{-L}^{L} f(x)\sin(k\pi x/L)\,dx\right] \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)(\cos(k\pi x/L) \mp i\sin(kx))\,dx \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)e^{\mp ik\pi x/L}\,dx
\end{aligned} \tag{2.14}$$

and

$$c_0 = \tfrac{1}{2}a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)e^{0}\,dx\,.$$

The case $k = 0$ is therefore already covered by (2.14) and the formula for the complex Fourier coefficients is:

Complex Fourier Series Coefficients

$$\boxed{c_k = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik\pi x/L}\,dx \qquad \text{for} \quad k = -N, ..., N} \tag{2.15}$$

In this formulation $f(x)$ may even be a complex function of a real variable x.

An alternative way to establish the complex Fourier series, is to consider the set of functions

$$\{\phi_k(x) = e^{ik\pi x/L}, \quad k \in \mathbb{Z}\}$$

and show that they are orthogonal on the interval $[-L, L]$. The result is as follows,

$$\int_{-L}^{L} \overline{\phi_k(x)}\phi_j(x)\,dx = \begin{cases} 0, & k \neq j, \\ 2L, & k = j. \end{cases} \tag{2.16}$$

The formula for the Fourier approximation of $f(x)$ on $[-L, L]$ follows from (2.16) and (1.23). (Note however that in Paragraph 1.7.2 only real basis functions were considered.)

EXAMPLE 2.3

Find the complex Fourier series of $f(x) = e^x$, on $[-2, 2]$.

$$\begin{aligned} c_k &= \frac{1}{2\times 2}\int_{-2}^{2} e^x e^{-ik\pi x/2}\, dx \\ &= \frac{1}{4}\left.\frac{e^{x(1-ik\pi/2)}}{(1-ik\pi/2)}\right|_{-2}^{2} \\ &= \frac{\sinh(2-ik\pi)}{2-ik\pi} \end{aligned}$$

and the Fourier series is

$$f_\infty(x) = \sum_{k=-\infty}^{\infty} \frac{\sinh(2-ik\pi)}{2-ik\pi} e^{ik\pi x/2}. \tag{2.17}$$

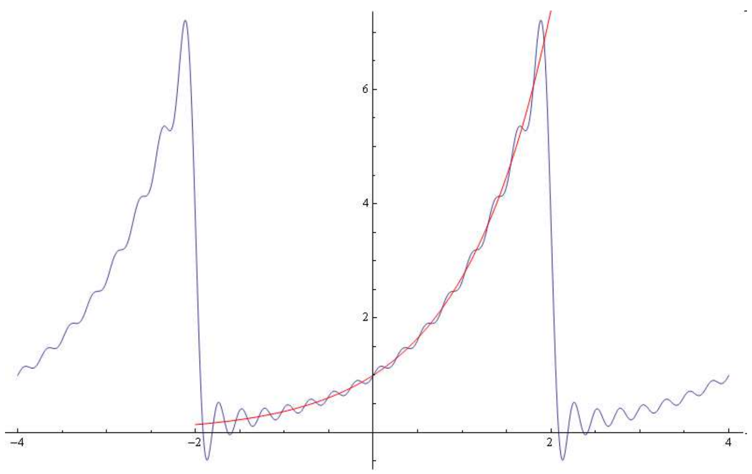


Figure 2.4: The truncated Fourier series with $N = 15$.

Although the Fourier approximation (2.17) might seem complex, this function is in fact real as should be the case given that $f(x)$ is a real function. The Fourier approximation (2.17) may be expanded and then simplified, yielding:

$$f_\infty(x) = \frac{\sinh(2)}{2} + \left(e^2 - e^{-2}\right)\sum_{k=1}^{\infty}\frac{(-1)^k}{4+k^2\pi^2}\left[2\cos\left(\frac{k\pi x}{2}\right) - k\pi\sin\left(\frac{k\pi x}{2}\right)\right]. \tag{2.18}$$

CHECK BOX 2.1

Check for yourself that (2.17) and (2.18) are indeed equivalent.

2.5 Half range and quarter range series

For a function defined on $x \in [-L, L]$ (the "full interval") the cos+sin Fourier series is straightforward. One must calculate the a_k's and the b_k's in the normal way (using (2.9)).

However, in many applications a function $f(x)$ is supplied on only some part of the interval, and it is required that some of the Fourier coefficients must be zero. This means that the function must be *completed* by using copies or reflections of $f(x)$ to "fill up" the rest of the interval. The most important series of this type are the so-called *half range series* and *quarter range series.*

2.5.1 Fourier half range series

The *half range series* is simple: The function $f(x)$ is given on $x \in [0, L]$, and the Fourier series must have either only sine terms, or only cosine terms.

2.5.1.1 Only sine terms in the series

The function is given on $x \in [0, L]$, and

$$f(x) = \sum_{k=1}^{\infty} b_k \sin(k\pi x/L).$$

Since only sine terms occur in the series, it is the series of an *odd* function. The function $f(x)$ must therefore be completed on the interval $[-L, L]$ as an odd function, i.e.

$$f_{completed}(x) = \begin{cases} f(x), & x \in [0, L), \\ -f(-x), & x \in [-L, 0). \end{cases}$$

Figure 2.5 shows $f(x)$ as well as $f_{completed}(x)$. Just note that it is not necessary to integrate over the whole interval. Because $f_{completed}$ is odd, we have

$$b_k = \frac{2}{L}\int_0^L f(x)\sin(k\pi x/L)\,dx\,.$$

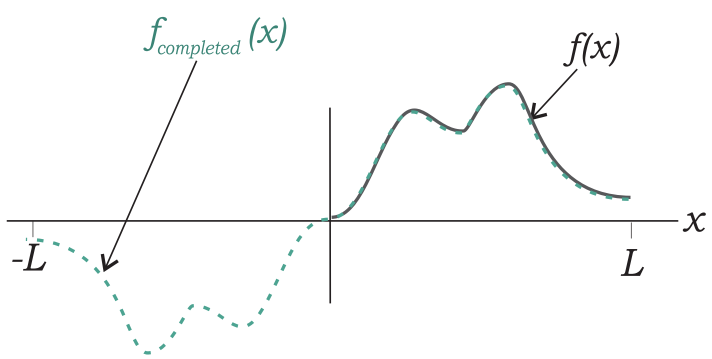


Figure 2.5: An odd completion of $f(x)$.

2.5.1.2 Only cosine terms in the series

The function is given on $x \in [0, L]$, and

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x/L).$$

Since only cosine terms occur in the series, it is the series of an *even* function. The function $f(x)$ must therefore be completed on the interval $[-L, L]$ as an even function, i.e.

$$f_{completed}(x) = \begin{cases} f(x), & x \in [0, L), \\ f(-x), & x \in [-L, 0). \end{cases}$$

Figure 2.6 shows $f(x)$ and $f_{completed}(x)$. Once again only integration over $[0, L]$ is needed,

$$a_k = \frac{2}{L} \int_0^L f(x) \cos(k\pi x/L)\, dx\,.$$

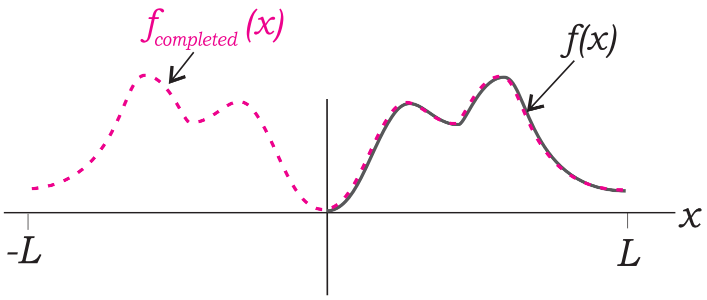


Figure 2.6: An even completion of $f(x)$.

2.5.2 Fourier quarter range series

With the *quarter range series* only one quarter of the function is supplied and it is required that its Fourier series has only sine or cosine terms and in addition there is the requirement that all the even coefficients should be zero (or alternatively, all the odd coefficients should be zero).

We shall illustrate this by considering an *even function* where only the *odd terms* remain. The function $f(x)$ is given on $x \in [0, L]$, and it is required that its Fourier series must be of the following form:

$$f(x) = \sum_{\substack{k=1 \\ k \,\text{odd}}}^{\infty} a_k \cos\left(\frac{k\pi x}{2L}\right).$$

Since only cosine terms occur in the series, it is the series of an even function. Since the arguments of the cosine functions are of the form $k\pi x/(2L)$, the function is periodic on $[-2L, 2L]$, but the function is given only over $[0, L]$. This means that only *a quarter* of the full function is supplied. Figure 2.7 shows the setup.

It is clear that $f(x)$ must be completed on the interval $[-2L, 2L]$ as an *even* function. Let the

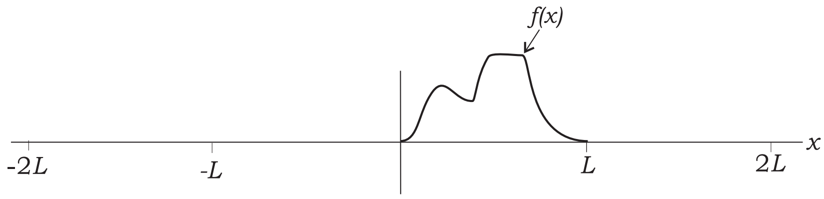


Figure 2.7: Only one quarter of the function is supplied

part of the completed function between $[L, 2L]$ be $g(x)$, so that $f_{completed}$ is given by

$$f_{completed}(x) = \begin{cases} g(-x), & x \in [-2L, -L), \\ f(-x), & x \in [-L, 0), \\ f(x), & x \in [0, L), \\ g(x), & x \in [L, 2L). \end{cases}$$

We need to find the relationship between $f(x)$ and $g(x)$.

The Fourier coefficients are

2, because an even function is integrated only over half the interval

$$a_k = \frac{2}{(2L)} \left[\int_0^L f(x) \cos\left(\frac{k\pi x}{2L}\right) dx + \int_L^{2L} g(x) \cos\left(\frac{k\pi x}{2L}\right) dx \right]. \tag{2.19}$$

(2L), because the full period is now $4L$

Consider the second integral

$$X = \int_L^{2L} g(x) \cos\left(\frac{k\pi x}{2L}\right) dx.$$

In order to convert the limits to $[0, L]$, we substitute $x = 2L - y$, into X then

$$\begin{aligned} X &= \int_L^0 g(2L-y) \cos\left(\frac{k\pi}{2L}(2L-y)\right)(-dy) \\ &= \int_0^L g(2L-y) \cos\left(\frac{k\pi}{2L}(2L-y)\right) dy \\ &= \int_0^L g(2L-y) \cos\left(\frac{-k\pi y}{2L} + k\pi\right) dy \\ &= \int_0^L g(2L-y) \left[\cos\left(\frac{-k\pi y}{2L}\right)\cos(k\pi) - \sin\left(\frac{-k\pi y}{2L}\right)\sin(k\pi)\right] dy \\ &= (-1)^k \int_0^L g(2L-y) \cos\left(\frac{k\pi y}{2L}\right) dy \end{aligned}$$

... using x again as integration variable ...

$$= (-1)^k \int_0^L g(2L-x) \cos\left(\frac{k\pi x}{2L}\right) dx\,. \tag{2.20}$$

Substituting (2.20) back into (2.19) yields

$$a_k = \frac{1}{L}\int_0^L \Big(f(x) + (-1)^k g(2L-x)\Big)\cos\left(\frac{k\pi x}{2L}\right)dx\,.$$

We now impose the requirement that all a_k must be zero for even k, i.e.

$$0 = \frac{1}{L}\int_0^L \Big(f(x) + g(2L-x)\Big)\cos\left(\frac{k\pi x}{2L}\right)dx\,.$$

This can only be valid for all k, if

$$f(x) + g(2L-x) = 0, \qquad \text{for} \quad x \in [0, L),$$

that is,

$$g(x) = -f(2L-x), \qquad \text{for} \quad x \in [L, 2L).$$

Then $g(x)$ is just $f(x)$ that is flipped about the y-axis, as well as about the x-axis, and shifted to the interval $[L, 2L]$.

Figure 2.8 shows $f(x)$ and $f_{completed}(x)$.

Once again only integration over $[0, L]$ is needed, since for odd k, we have

$$a_k = \frac{1}{L}\int_0^L \Big(f(x) - g(2L-x)\Big)\cos\left(\frac{k\pi x}{2L}\right)dx,$$

and substituting $g(2L-x) = -f(x)$, gives

$$a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi x}{2L}\right)dx.$$

There are four types of quarter range series. The formulae for each may be derived as in the above example. The formulae are listed in the table below and typical graphs are shown in Figure 2.9.

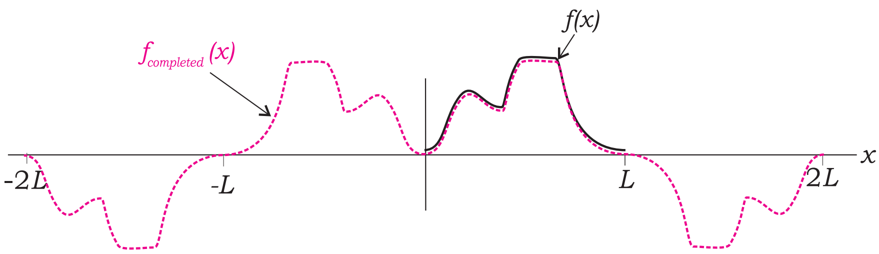


Figure 2.8: The completion of $f(x)$ on $[-2L, 2L]$.

Table 2.1: The four types of quarter range series shown in Figure 2.9.

(A)	$f_{cmpl}(x) = \begin{cases} -f(2L+x), & x \in [-2L,-L) \\ +f(-x), & x \in [-L,0) \\ +f(x), & x \in [0,L) \\ -f(2L-x), & x \in [L,2L) \end{cases}$	$f(x) = a_1 \cos\left(\frac{\pi x}{2L}\right) + a_3 \cos\left(\frac{3\pi x}{2L}\right) + a_5 \cos\left(\frac{5\pi x}{2L}\right) + \ldots$
(B)	$f_{cmpl}(x) = \begin{cases} +f(2L+x), & x \in [-2L,-L) \\ +f(-x), & x \in [-L,0) \\ +f(x), & x \in [0,L) \\ +f(2L-x), & x \in [L,2L) \end{cases}$	$f(x) = \frac{1}{2}a_0 + a_2 \cos\left(\frac{2\pi x}{2L}\right) + a_4 \cos\left(\frac{4\pi x}{2L}\right) + a_6 \cos\left(\frac{6\pi x}{2L}\right) + \ldots$
(C)	$f_{cmpl}(x) = \begin{cases} +f(2L+x), & x \in [-2L,-L) \\ -f(-x), & x \in [-L,0) \\ +f(x), & x \in [0,L) \\ -f(2L-x), & x \in [L,2L) \end{cases}$	$f(x) = b_2 \sin\left(\frac{2\pi x}{2L}\right) + b_4 \sin\left(\frac{4\pi x}{2L}\right) + b_6 \sin\left(\frac{6\pi x}{2L}\right) + \ldots$
(D)	$f_{cmpl}(x) = \begin{cases} -f(2L+x), & x \in [-2L,-L) \\ -f(-x), & x \in [-L,0) \\ +f(x), & x \in [0,L) \\ +f(2L-x), & x \in [L,2L) \end{cases}$	$f(x) = b_1 \sin\left(\frac{\pi x}{2L}\right) + b_3 \sin\left(\frac{3\pi x}{2L}\right) + b_5 \sin\left(\frac{5\pi x}{2L}\right) + \ldots$

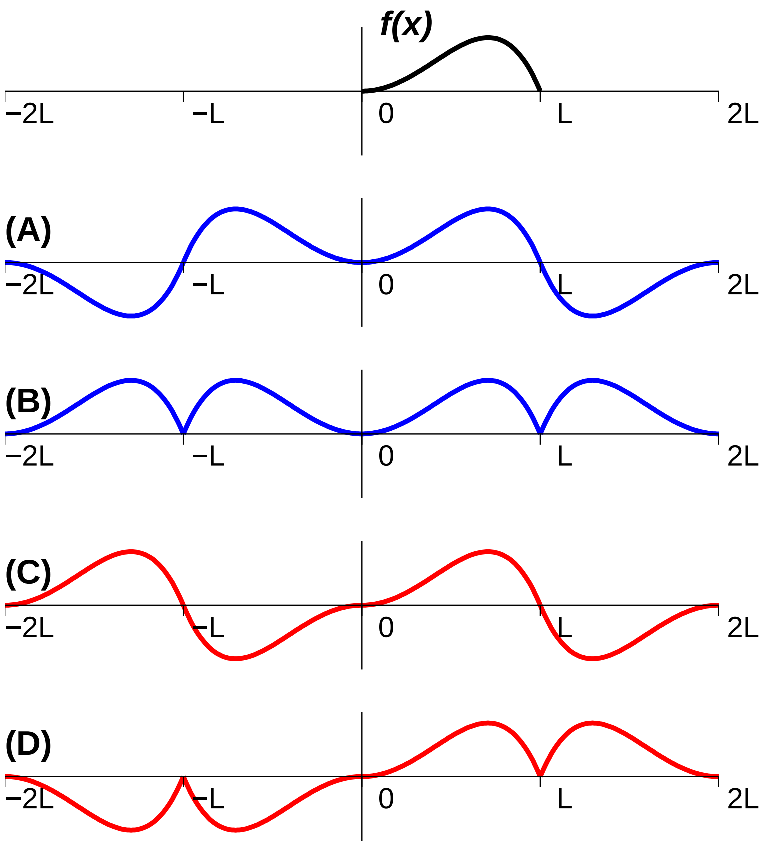


Figure 2.9: Examples of the four types of quarter series on $[-2L, 2L]$ using the same $f(x)$.

PROBLEM SET 2: The Fourier series

1. **(a)** Without the help of a symbolic mathematical computation program (e.g. MATHEMATICA), find the sin+cos Fourier series of

$$f(x) = 1 - |x|, \quad x \in [-1, 1)$$

by hand.

(b) Find a formula (of infinitely many terms) for π^2 by evaluating the above Fourier series at $x = 0$. Also check the formula numerically by summing the first ten thousand terms.

2. Do the following by hand (without the help of a symbolic mathematical computation program, e.g. MATHEMATICA).

Use MATLAB (or any other suitable software) to draw graphs of the Fourier approximation on the function (specify a value for L if applicable). Use enough terms to render a sensible representation (but not so much that one cannot differentiate between the function and the Fourier series).

Find the sin+cos Fourier series of

(a)

$$f(x) = \begin{cases} 1, & x \in (-2, -1) \\ 0, & x \in (-1, 1) \\ -1, & x \in (1, 2) \end{cases}$$

(b)

$$f(x) = \begin{cases} 1, & x \in (-2, -1) \\ 0, & x \in (-1, 1) \\ 1, & x \in (1, 2) \end{cases}$$

(c)

$$f(x) = \begin{cases} 0, & x \in (-2, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, 2) \end{cases}$$

(d)

$$f(x) = \begin{cases} 0, & x \in (-L, L/2) \\ 1, & x \in (L/2, L) \end{cases}$$

(e)

$$f(x) = |x|, \quad x \in [-\pi, \pi)$$

(f)

$$f(x) = x, \quad x \in (-\pi, \pi)$$

(g)

$$f(x) = 1 - x^2, \quad x \in [-1, 1)$$

(h)

$$f(x) = |\sin(x)|, \quad x \in [-\pi, \pi)$$

You may check your answer with e.g. MATHEMATICA.

Be careful: There might be special cases for k where the denominator of a term in the series is zero. These special cases for k must be integrated separately.

3. Write down two different functions of your own choice (defined on a finite interval) with the following properties:

- $f(x)$, whose periodic extension is discontinuous (has a jump somewhere),
- $g(x)$, whose periodic extension is continuous, but its first derivative is discontinuous.

Do not use any functions given in this problem set, i.e. *create your own functions.*

Calculate the sin+cos Fourier series of $f(x)$ by hand and redo it in a symbolic computation program, e.g. MATHEMATICA. Hand in both the handwritten work as well as a print out of your computer work.

Find the Fourier series of $g(x)$ by only using a symbolic computation program, e.g. MATHEMATICA. Illustrate the function and its Fourier approximation with suitable plots.

4. Do the following by hand (without the help of a symbolic mathematical computation program, e.g. MATHEMATICA).

Use MATLAB (or any other suitable software) to draw graphs of the Fourier approximation on the function (specify a value for L if applicable). Use enough terms to render a sensible representation (but not so much that one cannot differentiate between the function and the complex Fourier series).

Find the complex Fourier series of

(a)

$$f(x) = \begin{cases} 0, & x \in [-\pi, 0) \\ x, & x \in (0, \pi) \end{cases}$$

(b)

$$f(x) = \begin{cases} 0, & x \in (-L, 0] \\ xe^{-x}, & x \in (0, L) \end{cases}$$

(c)

$$f(x) = e^{-|x|}, \quad x \in [-1, 1)$$

You may check your answer with e.g. MATHEMATICA.

Be careful: There might be special cases for k where the denominator of a term in the series is zero. These special cases for k must be integrated separately.

5. Solve a complex Fourier series (do not use a function given in this problem set, i.e. *create your own function*) in any other suitable symbolic mathematical computation program, e.g. MATHEMATICA, and illustrate its convergence graphically.

Chapter 3

The Fourier transform

3.1 Introduction

"The formula for the Fourier coefficients links a periodic function $f(x)$ to an infinite set of coefficients c_k, and the formula for the Fourier series transforms these coefficients back to $f(x)$. There is therefore the notion of going 'back and forth' between $f(x)$ and c_k. We may consider the c_k to be a 'transform' of $f(x)$, and $f(x)$ to be the 'inverse transform' of c_k.

The continuous Fourier transform extends this notion to functions $f(x)$ that are not periodic."

We restate the Fourier formulae for a complex series here:

$$c(k) = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-ik\pi x/L} \, dx \,, \qquad \text{for } k \in \mathbb{Z}, \tag{3.1}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c(k) e^{ik\pi x/L} \,, \qquad \text{for } x \in \mathbb{R}. \tag{3.2}$$

Note that we have replaced c_k with a new notation $c(k)$, implying that c is a function of k – it is in fact a discrete function defined on $k \in \mathbb{Z}$.

It will be shown in the next section that the formulae for the Fourier transform (FT) and the inverse Fourier transform (IFT), can be derived from (3.1) and (3.2).

If $f(x)$ is defined on $x \in (-\infty, \infty)$, then $f(x)$ is square integrable if $\int_{-\infty}^{\infty} |f(x)|^2 dx^*$ is finite (which can only hold if $\lim_{x \to \pm\infty} f(x) = 0$). The FT of $f(x)$ is then given by $F(\xi)$ below. $F(\xi)$ is then also square integrable and $f(x)$ can be recovered from $F(\xi)$ by using the formula for the IFT.

Fourier Transform

$$\boxed{F(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx} \tag{3.3}$$

Inverse Fourier Transform

$$\boxed{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) e^{i\xi x} d\xi} \tag{3.4}$$

$^*|f(x)|^2 = \overline{f(x)} f(x)$

Note that f is a function of x, while F is a function of ξ. In practical applications x may be measured for example in m, then ξ is measured in m^{-1}, or x may be measured for example in s, then ξ is measured in s^{-1}, i.e. Hz. The variable x is referred to as "physical space" while the variable ξ is referred to a "frequency space". Both $f(x)$ and $F(\xi)$ represent the same function: $f(x)$ is a physical space representation and $F(\xi)$ is a frequency space representation.

We shall employ the notation that functions will be written with lowercase Latin symbols, and their corresponding FTs will be denoted by the corresponding uppercase letters. We shall further use the notation that the corresponding frequency of x is given by its Greek counterpart, ξ, and, likewise, the corresponding frequency of y is given by η. For example, the FT of $g(x)$ is $G(\xi)$ and the FT of $r(y)$ is $R(\eta)$.

We shall now show some examples.

EXAMPLE 3.1

Find the FT of $f(x)$, where

$$f(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| > 1. \end{cases}$$

$$\begin{aligned} F(\xi) &= \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \\ &= \int_{-1}^{1} 1 \cdot e^{-i\xi x} dx \\ &= \left[\frac{e^{-i\xi x}}{-i\xi} \right]_{-1}^{1} \\ &= \frac{2}{\xi} \left[\frac{e^{-i\xi} - e^{i\xi}}{-2i} \right] \\ &= \frac{2\sin(\xi)}{\xi}. \end{aligned}$$

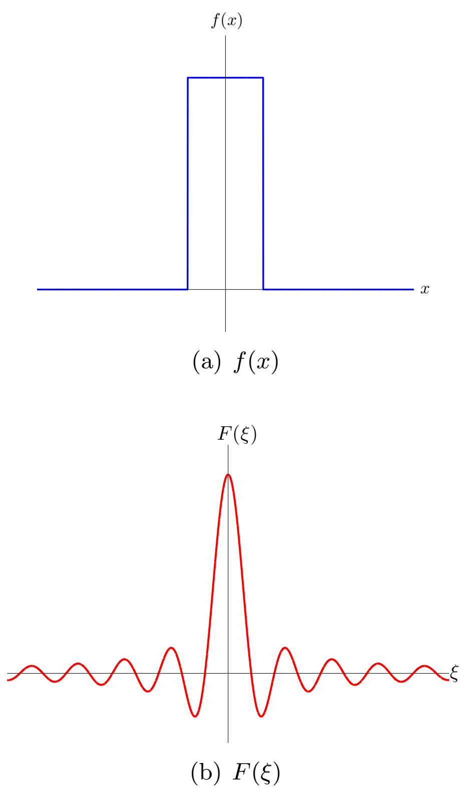


Figure 3.1: The function $f(x)$ of Example 3.1 and its Fourier transform, $F(\xi)$.

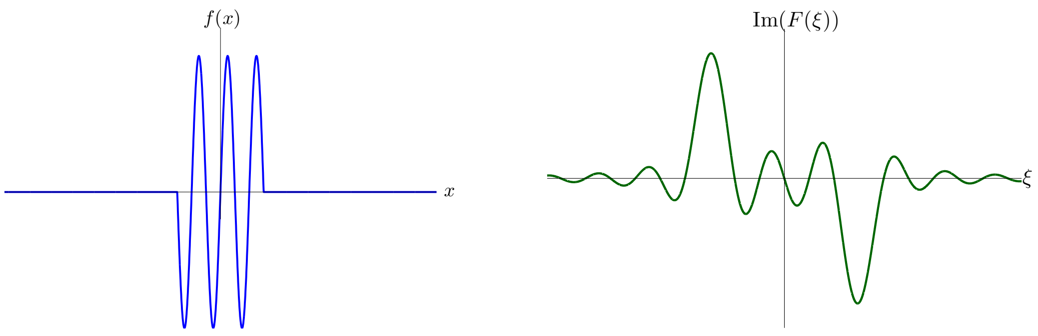
EXAMPLE 3.2

Find the FT of

$$f(x) = \begin{cases} \sin(\omega x) & |x| < 1, \\ 0 & |x| \geq 1, \end{cases}$$

where ω is a constant.

$$\begin{aligned}
F(\xi) &= \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \\
&= \int_{-1}^{1} \sin(\omega x) \cdot e^{-i\xi x} dx \\
&= \int_{-1}^{1} \frac{e^{i\omega x} - e^{-i\omega x}}{2i} e^{-i\xi x} dx \\
&= \frac{1}{2i} \left[\frac{e^{i(\omega-\xi)x}}{i(\omega-\xi)} - \frac{e^{i(-\omega-\xi)x}}{-i(\omega+\xi)} \right]_{-1}^{1} \\
&= -i \left[\frac{e^{i(\omega-\xi)} - e^{-i(\omega-\xi)}}{2i(\omega-\xi)} + \frac{e^{-i(\omega+\xi)} - e^{i(\omega+\xi)}}{2i(\omega+\xi)} \right] \\
&= i \left[\frac{\sin(\omega+\xi)}{(\omega+\xi)} - \frac{\sin(\omega-\xi)}{(\omega-\xi)} \right].
\end{aligned}$$



(a) $f(x)$ where $\omega = 3\pi$

(b) $F(\xi)$ where $\omega = 3\pi$

Figure 3.2: The function $f(x)$ of Example 3.2 and its Fourier transform, $F(\xi)$.

3.2 Derivation of the Fourier transform from the Fourier series

"The Fourier transform is a continuous analogue of the semi-discrete Fourier series. We shall now show how the Fourier series and the Fourier transform are related and then illustrate this relation by means of an example."

Consider the formulae (3.1) and (3.2) for the complex Fourier series, again:

$$c(k) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik\pi x/L}\,dx, \qquad \text{for } k \in \mathbb{Z}, \tag{3.5}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c(k)e^{ik\pi x/L}, \qquad \text{for } x \in \mathbb{R}. \tag{3.6}$$

The Fourier series approximates the periodic continuation of a function $f(x)$ that is defined on $x \in (-L, L)$, whereas the Fourier transform approximates a function $f(x)$ that is defined on $x \in (-\infty, \infty)$. Therefore, to convert from a Fourier series to a Fourier transform, an infinite "window" must be enforced on the Fourier series, i.e. L must be increased to infinity. From (3.5), one would expect from the $1/L$ scaling of the function and its frequency, that $c(k)$ and the frequency of $c(k)$ will decrease (i.e. shrinks vertically and gets wider horizontally) with increasing L. We now try to retain the shape of the function as $L \to \infty$ by "scaling back" the shrinking and widening effect.

Let

$$\xi_k = k\Delta\xi\,, \quad k \in \mathbb{Z}, \tag{3.7}$$

where

$$\Delta\xi = \frac{\pi}{L}\,. \tag{3.8}$$

Therefore

$$\xi_k = \frac{k\pi}{L}\,, \quad k \in \mathbb{Z}\,. \tag{3.9}$$

Multiply (3.5) by $2L$, yields

$$F(\xi_k) = \int_{-L}^{L} f(x)e^{-ik\pi x/L}\,dx = \int_{-L}^{L} f(x)e^{-i\xi_k x}\,dx\,, \tag{3.10}$$

where

$$F(\xi_k) = 2Lc(k)\,. \tag{3.11}$$

The reason for including a factor 2 is simply to end up with a unit coefficient as required by the format of the Fourier transform, (3.3).

Now, let $L \to \infty$, then $\Delta\xi$ tends to zero and ξ_k becomes continuous and (3.10) turns into

$$F(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx\,. \tag{3.12}$$

This is the formula for the Fourier transform of $f(x)$ (compare to (3.3)).

How can $f(x)$ be recovered from $F(\xi)$?

Take (3.6), substitute (3.9), and replace $c(k)$ by $F(\xi_k)/(2L)$:

$$f(x) \;=\; \sum_{k=-\infty}^{\infty} \frac{1}{2L} F(\xi_k) e^{i\xi_k x}. \tag{3.13}$$

Note that, from (3.8), $\dfrac{1}{2L} = \dfrac{\Delta\xi}{2\pi}$, yielding

$$f(x) \;=\; \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} F(\xi_k) e^{i\xi_k x} \Delta\xi\,, \tag{3.14}$$

that may be expressed as integration by using the rectangle rule. Taking the limit as $L \to \infty$, yields

$$f(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) e^{i\xi x}\, d\xi \tag{3.15}$$

which is the formula for the inverse Fourier transform of $F(\xi)$ (compare to (3.4)).

It often happens that the integral in (3.15) cannot be solved analytically. Equation (3.15) is then an integral representation of $f(x)$, as will be discussed in Paragraph 3.3.

A specific example:

To clarify the theory behind the transition from the Fourier series coefficients (3.5) to the Fourier transform (3.12), we shall consider a particular example of a square wave pulse.

For $L > 1$ let

$$f(x) \;=\; \begin{cases} 0, & \text{if } x \in [-L, -1), \\ 1, & \text{if } x \in (-1, 1), \\ 0, & \text{if } x \in (1, L]. \end{cases} \tag{3.16}$$

The complex Fourier coefficients of $f(x)$ are given by

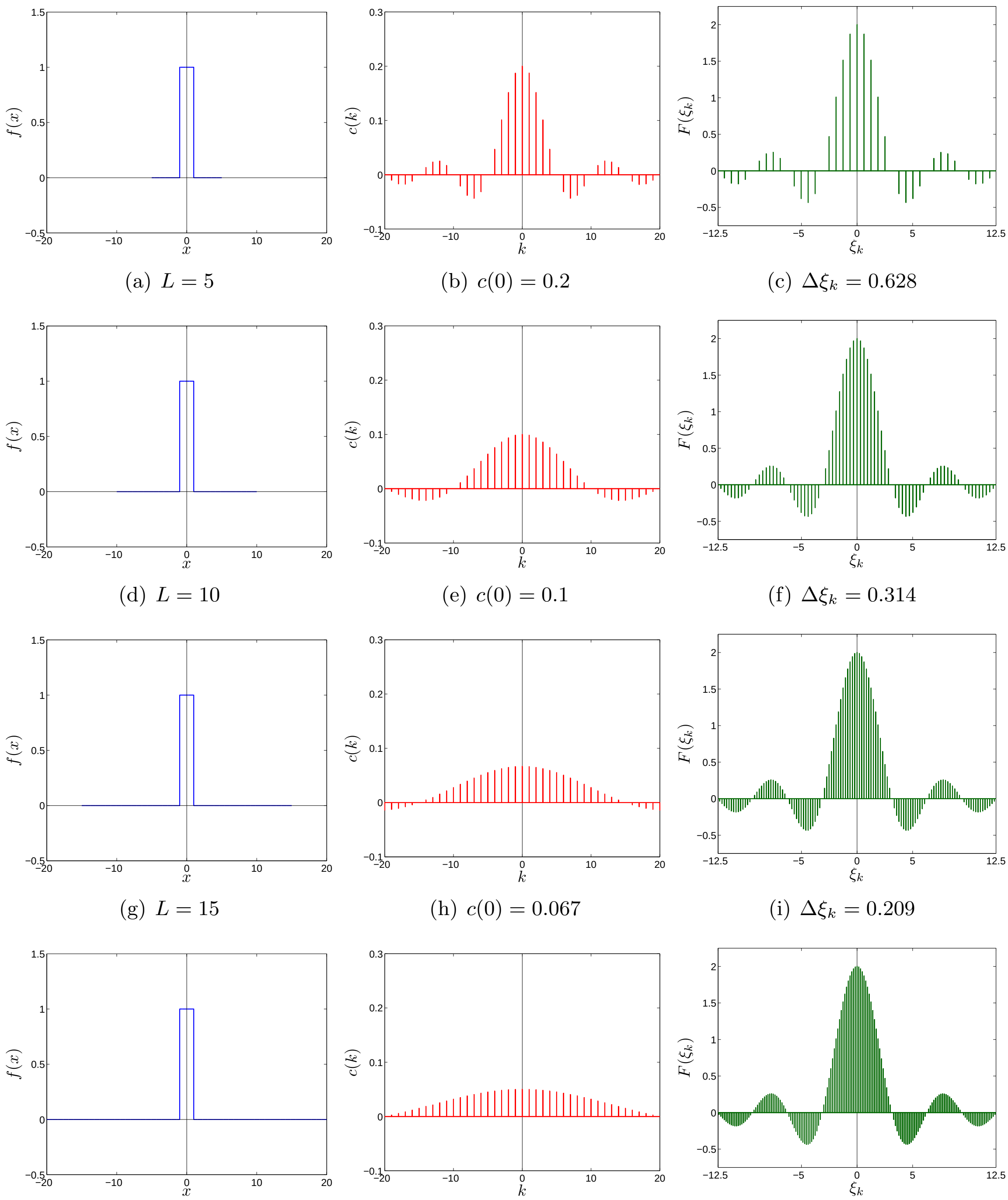
$$c(k) \;=\; \frac{1}{2L} \int_{-1}^{1} 1 \cdot e^{-i\pi k x/L}\, dx \;=\; \frac{1}{\pi k} \sin(\pi k/L). \tag{3.17}$$

The coefficients may also be expressed as

$$c(k) \;=\; \frac{1}{L} \left(\frac{\sin(\pi k/L)}{\pi k/L} \right) \tag{3.18}$$

which may be recognised as the sinc function with argument $(\pi k/L)$, and the function is scaled by $1/L$.

Figure 3.3 shows $f(x)$ in the first column and $c(k)$ in the second column for four values of L, viz. $L = 5$, $L = 10$, $L = 15$, and $L = 20$. As would be expected from the $1/L$ scaling, the function $c(k)$ shrinks vertically with increasing L. Furthermore, $c(k)$ gets wider horizontally with increasing L, which is expected from the scaling of the argument by π/L.



(a) $L = 5$ (b) $c(0) = 0.2$ (c) $\Delta\xi_k = 0.628$

(d) $L = 10$ (e) $c(0) = 0.1$ (f) $\Delta\xi_k = 0.314$

(g) $L = 15$ (h) $c(0) = 0.067$ (i) $\Delta\xi_k = 0.209$

(j) $L = 20$ (k) $c(0) = 0.05$ (l) $\Delta\xi_k = 0.157$

Figure 3.3: The rectangular pulse function, its Fourier coefficient, and its scaled Fourier coefficient, for $L = 5, 10, 15, 20$.

In order to retain the basic function $\text{sinc}(\pi k/L)$, as $L \to \infty$, the shrinking and widening effect may be "scaled back" by imposing (3.9) and (3.11), yielding

$$F(\xi_k) \;=\; 2\left(\frac{\sin(\xi_k)}{\xi_k}\right),$$

as we expected (see Example 3.1).

Now, as $L \to \infty$, $F(\xi_k)$ retains the basic shape of the sinc function. However, note that the increment $\Delta\xi$ decreases to zero as L increases to infinity. This means that as $L \to \infty$, ξ_k becomes a continuous variable. The third column in Figure 3.3 shows $F(\xi_k)$ for the same four values of L. Note that the "shape" of F stays constant but the spacing between consecutive values of ξ_k becomes smaller.

3.3 Integral representations

Since the IFT of the FT of a function $f(x)$, is equal to $f(x)$, we may express $f(x)$ as the IFT of the FT of $f(x)$. Let us try that with the function in Example 3.1.

Consider the function

$$f(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

From Example 3.1 we know that the FT of $f(x)$ is

$$F(\xi) = \frac{2\sin(\xi)}{\xi}.$$

Give an integral representation of $f(x)$.

The function, $f(x)$, may also be expressed as the IFT of the FT:

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(\xi)}{\xi} e^{ix\xi}\, d\xi \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\xi)}{\xi}\cos(x\xi)\, d\xi + \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\xi)}{\xi}\sin(x\xi)\, d\xi \end{aligned}$$

... but the second integral is the integral of an odd function over a symmetric interval, and is therefore zero ...

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\xi)}{\xi}\cos(x\xi)\, d\xi.$$

In other words:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\xi)}{\xi}\cos(x\xi)\, d\xi = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

This gives a new way to express the rectangular pulse function. It is called an *integral representation* of a piecewise defined function. In order to describe the rectangular pulse function mathematically, we may either use the piecewise definition (the right hand side), or define the function as a single expression containing an infinite integral (the left hand side).

3.4 Properties of the Fourier transform

> *"Why would anyone want to compute the Fourier transform of a function? It is because some of the properties of the FT allow us to do some operations much easier in the frequency domain than in the physical domain. After the operation has been completed we can return to physical space by applying the IFT.*
>
> *We shall now list many of the properties of the FT (most without derivation)."*

We shall use the notation

$$\mathcal{F}\left[f(x)\right]_{x\to\xi} = F(\xi),$$

which means *the Fourier transform of f is F and the physical variable x is associated with the frequency variable ξ.*

The IFT will be denoted by

$$\mathcal{F}^{-1}\left[F(\xi)\right]_{\xi\to x} = f(x),$$

which means *the inverse Fourier transform of F is f and the frequency variable ξ is associated with the physical variable x.*

3.4.1 Parity and complex values

It can be shown that for a function with definite parity, the parity is retained in the transition from physical to frequency space. For odd functions, a real function becomes imaginary in frequency space and vice versa, and, for even functions, a real (or imaginary) function stays real (or imaginary) in frequency space. This is summarised in Table 3.1. The Fourier transform of a function with no particular parity, is also without parity.

Table 3.1: The following is true of functions with definite parity which is real only, or imaginary only.

$f(x)$	$F(\xi)$
real & even	real & even
real & odd	imaginary & odd
imaginary & even	imaginary & even
imaginary & odd	real & odd

In Tables 3.2 and 3.3 graphs of some square integrable functions are shown together with their Fourier transforms. We have chosen functions that are all real and have definite parity. Their corresponding FTs then are either real or imaginary and also have definite parity.

Table 3.2: Even functions with their Fourier transforms.

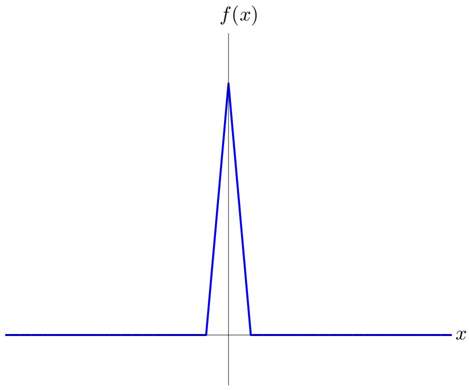
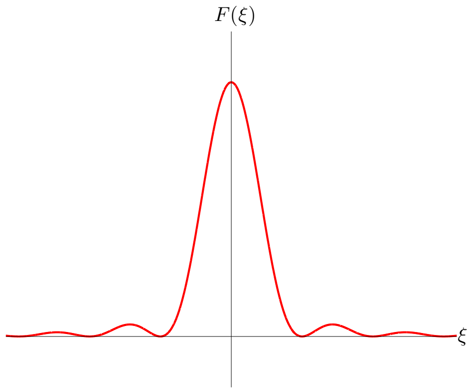
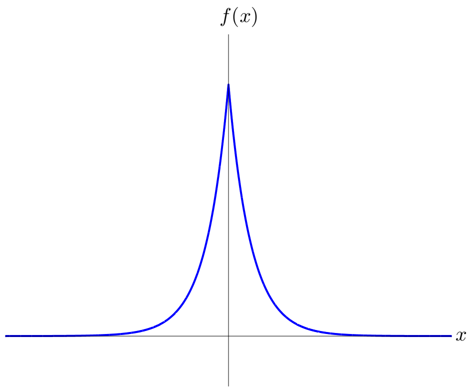
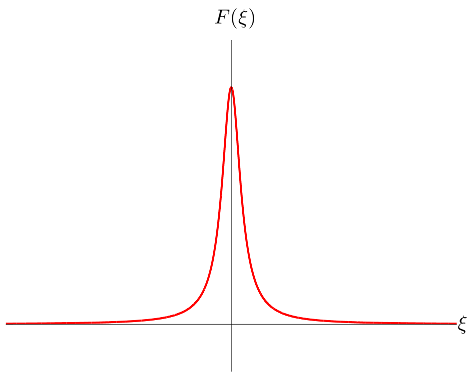
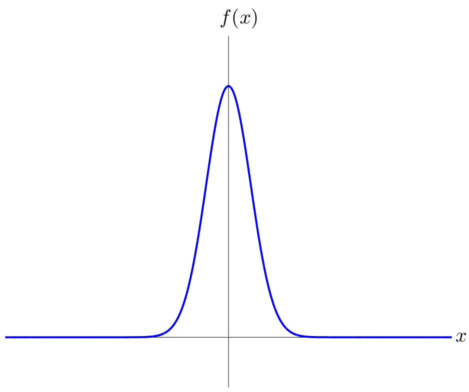
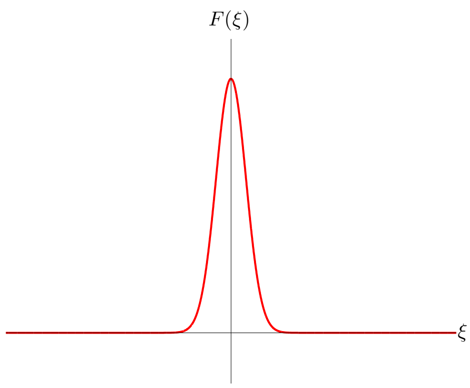
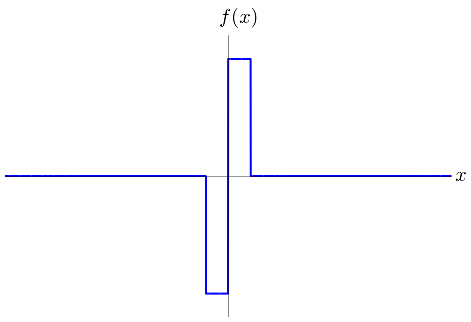
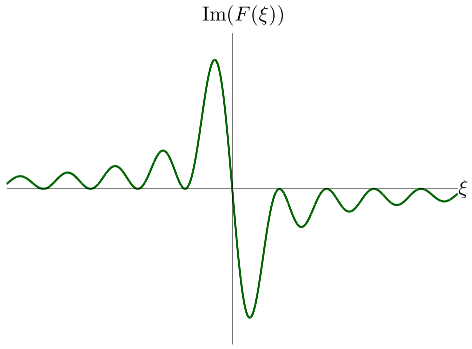
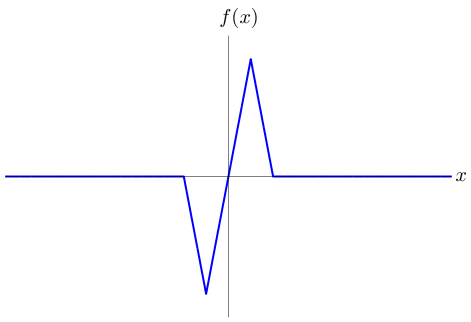
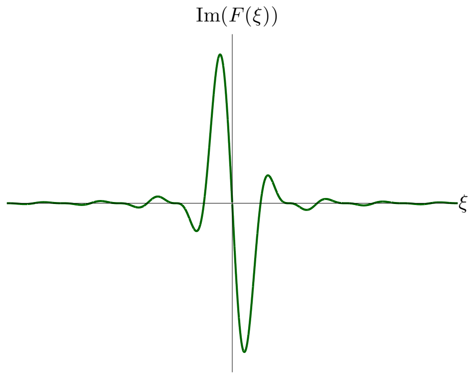
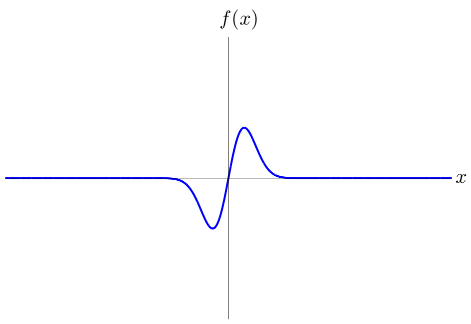
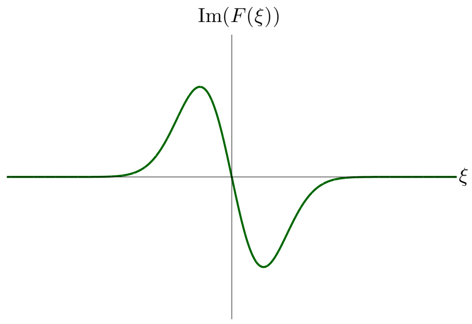
	Function	**Fourier Transform**
(A)	**Roof-top function** $f(x) = \begin{cases} 1-\lvert x\rvert, & x \in [-1,1] \\ 0, & \text{otherwise} \end{cases}$ 	$F(\xi) = \dfrac{2(1-\cos\xi)}{\xi^2}$ 
(B)	**Exponential decay** $f(x) = e^{-\lvert x\rvert}$ 	$F(\xi) = \dfrac{2}{1+\xi^2}$ 
(C)	**Gauss function** $f(x) = e^{-\frac{1}{2}x^2}$ 	$F(\xi) = \sqrt{2\pi}e^{-\frac{1}{2}\xi^2}$ 

Table 3.3: Odd functions with their Fourier transforms.

	Function	**Fourier Transform**
(D)	**Double square wave function** $f(x) = \begin{cases} -1, & x \in [-1,0] \\ 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$ 	$\mathrm{Im}(F(\xi)) = \dfrac{2(\cos\xi - 1)}{\xi}$ 
(E)	**Double roof-top function** $f(x) = \begin{cases} -2-x, & x \in [-2,-1] \\ x, & x \in [-1,1] \\ 2-x, & x \in [1,2] \\ 0, & \text{otherwise} \end{cases}$ 	$\mathrm{Im}(F(\xi)) = \dfrac{4\sin\xi(\cos\xi - 1)}{\xi^2}$ 
(F)	**Scaled Gaussian derivative** $f(x) = xe^{-x^2}$ 	$\mathrm{Im}(F(\xi)) = -\frac{1}{2}\sqrt{\pi}\xi e^{-\xi^2/4}$ 

3.4.2 Linearity

$$\boxed{\mathcal{F}\Big[\alpha f(x) + \beta g(x)\Big]_{x\to\xi} = \alpha F(\xi) + \beta G(\xi)} \tag{3.19}$$

This is easily proven by using the linearity of integration.

3.4.3 Scaling the argument

$$\boxed{\mathcal{F}\Big[f(\beta x)\Big]_{x\to\xi} = \frac{1}{|\beta|}F(\xi/\beta)} \tag{3.20}$$

Physically this means that when a function is stretched wider, its Fourier transform becomes narrower and vice versa. It is this effect that is responsible for the *Heisenberg uncertainty principle* in physics, i.e. the more accurate the momentum of a quantum particle is determined, the less precisely its position can be known, and vice versa. This is because the momentum wave function of a particle is the FT of the position wave function.

3.4.4 Shifting the argument

$$\boxed{\mathcal{F}\Big[f(x+\omega)\Big]_{x\to\xi} = e^{i\omega\xi}F(\xi)} \tag{3.21}$$

This effect is often used in applications to classify functions regardless of their particular shift.

3.4.5 Differentiation

$$\boxed{\mathcal{F}\left[\frac{d^n f(x)}{dx^n}\right]_{x\to\xi} = (i\xi)^n F(\xi)} \tag{3.22}$$

This is perhaps one of the most useful properties of the Fourier transform. It means that differentiating n times in physical space is equivalent to multiplying the FT by the n-th power of $i\xi$ in frequency space.

This formula relates the n-th power of a function to its n-th derivative. For square integrable functions, this formulation provides a way to compute *fractional derivatives* of a function.

This property will be derived here only for the case $n = 1$. You may use induction to generalise this result for all positive integer n. You are reminded that square integrable functions have the property that $\lim_{x\to\pm\infty} f(x) = 0$. This is also true for any derivative of a square integrable function.

$$\begin{aligned}
\mathcal{F}\left[f'(x)\right]_{x\to\xi} &= \int_{-\infty}^{\infty} f'(x)e^{-ix\xi}\,dx \\
&= \left[f(x)e^{-ix\xi}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\frac{d}{dx}e^{-ix\xi}\,dx
\end{aligned}$$

... the first term is zero because $f(x)$ is square integrable ...

$$\begin{aligned}
&= -(-i\xi)\int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,dx \\
&= i\xi F(\xi)
\end{aligned}$$

3.5 The Gauss function

The function e^{-x^2} appears often in statistics and physics. It is called the Gauss function or just the *Gaussian* (and in statistics, a scaled version of it is called the normal distribution). Figure 3.4 shows this bell shaped function.

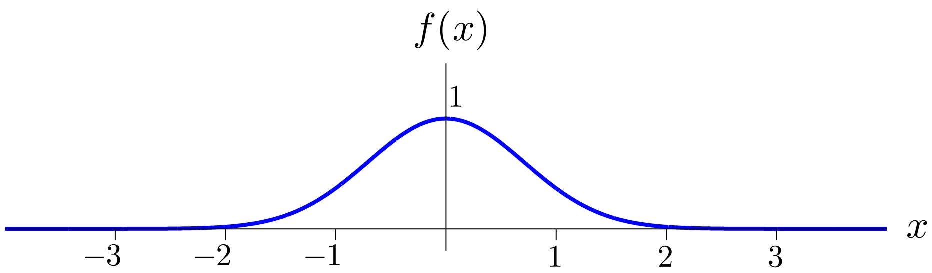


Figure 3.4: The Gauss function, $f(x) = e^{-x^2}$.

We shall now attempt to find the FT of this function. Let us first check the integral of the function i.e. $\int e^{-x^2} dx$. This integral is usually expressed in terms of a new special function called the *error-function*. However, if the limits of integration are infinite, the value of this integral can be found without the aid of the error function. The following special trick will be used.

Let

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx,$$

then

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy\,.$$

This is an area integral over the entire Cartesian plane. Now transform to polar coordinates: Let $r^2 = x^2 + y^2$, and $\tan\theta = y/x$, then the area element $dxdy$ is transformed to $rd\theta dr$ and the limits change to $r \in [0, \infty)$ and $\theta \in [0, 2\pi]$, yielding

$$\begin{aligned} I^2 &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\, dr\, d\theta \\ &= \int_0^{2\pi} \left[\frac{-e^{-r^2}}{2} \right]_0^{\infty} d\theta \\ &= \int_0^{2\pi} \tfrac{1}{2}\, d\theta \\ &= \pi\,. \end{aligned}$$

Therefore

$$I = \sqrt{\pi}\,.$$

In order to find the FT of the Gauss function, another special technique is used:

$$F(\xi) = \int_{-\infty}^{\infty} e^{-x^2} e^{-ix\xi}\, dx.$$

This integral cannot be found analytically, but let us use the following technique: Differentiate the function to ξ,

$$\begin{aligned} \frac{dF(\xi)}{d\xi} &= \int_{-\infty}^{\infty} -ixe^{-x^2} e^{-ix\xi}\, dx \\ &= \frac{i}{2} \left[e^{-x^2} e^{-ix\xi} \right]_{-\infty}^{\infty} - \frac{i}{2} \int_{-\infty}^{\infty} (-i\xi) e^{-x^2} e^{-ix\xi}\, dx \\ &= -\frac{\xi}{2} F(\xi)\,, \end{aligned}$$

which is a differential equation in ξ,

$$\frac{dF(\xi)}{d\xi} = -\frac{\xi}{2} F(\xi)\,.$$

The solution is:

$$F(\xi) = Ke^{-\xi^2/4}\,,$$

where K is an integration constant. However note that

$$F(0) = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\,,$$

therefore $K = \sqrt{\pi}$ and

$$F(\xi) = \sqrt{\pi} e^{-\xi^2/4}\,. \tag{3.23}$$

This is an interesting FT pair: The Gauss function is (apart from some scaling) its own Fourier transform!

3.6 The Cos and Sin transforms

"The Fourier transform is applied to functions defined on $x \in (-\infty, \infty)$, and therefore it can only be used in the solution of PDEs where the domain is $x \in (-\infty, \infty)$. Often the domain is rather $x \in [0, \infty)$, and we need to have a transform with appropriate derivative properties that acts on this domain. Both the Cos transform and the Sin transform can perform this role.

With both the Cos transform and the Sin transform, there is, however, an extra piece of information that is needed at the point $x = 0$ in order to apply their respective derivative properties. But, in a well-posed boundary condition on the domain $x \in [0, \infty)$, there is also an extra condition needed at the point $x = 0$, and this supplies the information needed by the Cos or Sin transform."

3.6.1 The Cos transform

We now introduce a new transform called the *Cos transform*, abbreviated with CosTF. It is actually not really new, it is just a variant of the FT.

We shall use the notation

$$\mathcal{C}[f(x)]_{x\to\xi}$$

to denote the CosTF of the function $f(x)$, where the frequency variable associated with x is ξ. We shall also often employ the shorter notation $F_C(\xi)$ to denote the CosTF of $f(x)$.

Consider an <u>*even real*</u> function $f(x)$ defined on $x \in (-\infty, \infty)$. Its Fourier transform is given by

$$\begin{aligned} F(\xi) &= \int_{-\infty}^{\infty} f(x) e^{-ix\xi}\,dx \\ &= \int_{-\infty}^{\infty} f(x)\cos(x\xi)\,dx - i\int_{-\infty}^{\infty} f(x)\sin(x\xi)\,dx \end{aligned}$$

... the latter integral disappears because $f(x)\sin(x\xi)$ is odd ...

$$= 2\int_0^{\infty} f(x)\cos(x\xi)\,dx.$$

Let us define the relationship between the CosTF and the FT as

$$F_C(\xi) = \frac{1}{2}F(\xi). \tag{3.24}$$

In other words

$$\boxed{\mathcal{C}[f(x)]_{x\to\xi} = \int_0^{\infty} f(x)\cos(x\xi)\,dx.} \tag{3.25}$$

Although the original $f(x)$ was defined over the entire real line, the CosTF can be viewed as a transform for functions only defined on $x \in [0, \infty)$. Note that $F_C(\xi)$ is an even function of ξ, and at this stage it may be defined for all $\xi \in (-\infty, \infty)$, even though x lies only in $[0, \infty)$.

In order to transform back, note that

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2F_C(\xi)e^{ix\xi}\,d\xi \\ &= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} 2F_C(\xi)\cos(x\xi)\,d\xi + i\int_{-\infty}^{\infty} 2F_C(\xi)\sin(x\xi)\,d\xi\right] \end{aligned}$$

... once again, the latter integral disappears because $F_C(\xi)\sin(x\xi)$ is odd ...

$$= \frac{2}{\pi}\int_0^{\infty} F_C(\xi)\cos(x\xi)\,d\xi\,.$$

Therefore the *inverse Cos transform*, abbreviated by ICosTF, is given by

$$\boxed{\mathcal{C}^{-1}[F_C(\xi)]_{\xi\to x} = \frac{2}{\pi}\int_0^{\infty} F_C(\xi)\cos(x\xi)\,d\xi.} \tag{3.26}$$

We now have a transform that operates on the region $[0,\infty)$, and this can be used to solve boundary value problems on the semi-infinite domain $x \in [0,\infty)$, $t \in [0,\infty)$.

3.6.2 The Sin transform

There is a similar transform that operates on the domain $x \in [0,\infty)$. It is called the *Sin transform*, and is abbreviated by SinTF. We shall likewise also use the $\mathcal{S}[f(x)]_{x\to\xi}$ to denote the SinTF of a function $f(x)$, and similarly, we shall also use the shorter notation $F_S(\xi)$ to denote the SinTF of a function $f(x)$.

The SinTF can be derived from the FT by considering an odd real function $g(x)$ defined on $x \in (-\infty,\infty)$. The Fourier transform of $g(x)$ is given by

$$\begin{aligned} G(\xi) &= \int_{-\infty}^{\infty} g(x)e^{-ix\xi}\,dx \\ &= \int_{-\infty}^{\infty} g(x)\cos(x\xi)\,dx - i\int_{-\infty}^{\infty} g(x)\sin(x\xi)\,dx \end{aligned}$$

... the first integral disappears because $g(x)\cos(x\xi)$ is odd ...

$$= -2i\int_0^{\infty} g(x)\sin(x\xi)\,dx$$

We now define the relationship between the SinTF and the FT as

$$G_S(\xi) = \frac{i}{2}G(\xi). \tag{3.27}$$

In other words

$$\boxed{\mathcal{S}[g(x)]_{x\to\xi} = \int_0^{\infty} g(x)\sin(x\xi)\,dx.} \tag{3.28}$$

Note that $G_S(\xi)$ is an odd function of ξ, because it is the imaginary part of the FT of a real odd function.

In order to transform back, note that

$$\begin{aligned}
g(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{i}G_S(\xi)e^{ix\xi}\,d\xi \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}-2iG_S(\xi)\cos(x\xi)\,d\xi + i\int_{-\infty}^{\infty}-2iG_S(\xi)\sin(x\xi)\,d\xi\right] \\
&\qquad \text{... once again, the first integral disappears because } G_S(\xi)\cos(x\xi) \text{ is odd ...} \\
&= \frac{2}{\pi}\int_0^{\infty}G_S(\xi)\sin(x\xi)\,d\xi.
\end{aligned}$$

Therefore the *inverse Sin transform*, abbreviated by ISinTF, is given by

$$\boxed{\mathcal{S}^{-1}[G_S(\xi)]_{\xi\to x} = \frac{2}{\pi}\int_0^{\infty}G_S(\xi)\sin(x\xi)\,d\xi.} \tag{3.29}$$

Both the CosTF and the SinTF are defined only for square integrable functions on the interval $[0,\infty)$.

3.6.3 Derivative properties of the Cos and Sin transforms

For our purposes, it is only the second derivative that is used in solving second order PDEs via transform techniques (Chapter 5). We shall therefore only consider the appropriate transform of the second derivative of a function here.

Consider

$$\begin{aligned}
\mathcal{C}[f''(x)]_{x\to\xi} &= \int_0^{\infty}f''(x)\cos(x\xi)dx \\
&= \Big[f'(x)\cos(x\xi)\Big]_0^{\infty} - \int_0^{\infty}f'(x)(-\xi)\sin(x\xi)dx \\
&= \Big[0 - f'(0)\Big] + \xi\int_0^{\infty}f'(x)\sin(x\xi)dx \\
&= -f'(0) + \xi\left(\Big[f(x)\sin(x\xi)\Big]_0^{\infty} - \int_0^{\infty}f(x)(\xi)\cos(x\xi)dx\right) \\
&= -f'(0) + \xi\Big(0 - f(0)\times 0 - \xi F_C(\xi)\Big) \\
&= -f'(0) - \xi^2F_C(\xi)\,.
\end{aligned}$$

Therefore

$$\boxed{\mathcal{C}[f''(x)]_{x\to\xi} = -f'(0) - \xi^2F_C(\xi)\,.} \tag{3.30}$$

There is a similar identity for the SinTF of a second derivative, and it is derived in similar manner. We shall not derive it here, but simply state it as,

$$\boxed{\mathcal{S}[f''(x)]_{x\to\xi} \;=\; \xi f(0) - \xi^2 F_S(\xi)\,.} \tag{3.31}$$

3.6.4 The CosTF and SinTF of some functions

It is often easier to find the FT of a function defined on $[0,\infty)$, but extended to $(-\infty,\infty)$ as even (in the case of the CosTF) or odd (in the case of the SinTF) and then to use (3.24) or (3.27) to find its appropriate Cos or Sin transform. We shall do a few examples below.

EXAMPLE 3.3

Find the CosTF of $f(x) = e^{-b^2x^2}$, where b is a fixed parameter.

Following from (3.23) and (3.20), FT of $f(x)$ is

$$F(\xi) \;=\; \frac{\sqrt{\pi}}{|b|}e^{-\xi^2/(4b^2)}\,. \tag{3.32}$$

Using (3.24), it then follows that the Cos transform is

$$F_C(\xi) \;=\; \frac{1}{2}F(\xi) \;=\; \frac{\sqrt{\pi}}{2\,|b|}e^{-\xi^2/(4b^2)}\,.$$

EXAMPLE 3.4

Find the ICosTF of

$$F_C(\xi) = e^{-c^2\xi^2}, \qquad x \in [0,\infty)$$

where c is a fixed parameter.

Once again we shall make use of what we already know of the Fourier transform to help us. From Example 3.3, (3.32), and the definition for the FT and the IFT ((3.3) and (3.4)), we know that:

$$G(\eta) \;=\; \int_{-\infty}^{\infty} e^{-b^2y^2}e^{-iy\eta}\,dy \;=\; \frac{\sqrt{\pi}}{|b|}e^{-\eta^2/(4b^2)} \tag{3.33}$$

and that

$$g(y) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sqrt{\pi}}{|b|}e^{-\eta^2/(4b^2)}\right)e^{iy\eta}\,d\eta \;=\; e^{-b^2y^2}\,. \tag{3.34}$$

Following from (3.29), we want to find

$$f(x) = \frac{2}{\pi} \int_0^\infty e^{-c^2\xi^2} \cos(x\xi)\, d\xi\,. \tag{3.35}$$

Our objective is to rewrite this equation such that it has the same form as either (3.33) or (3.34), because their solutions are known.

Since the integrand in (3.35) is an even function, this equation may be rewritten as

$$f(x) = \frac{1}{\pi} \int_{-\infty}^\infty e^{-c^2\xi^2} \cos(x\xi)\, d\xi\,, \tag{3.36}$$

and, since the integral of an odd function over $(-\infty, \infty)$ is zero, we may add $\left(e^{-c^2\xi^2} i \sin(x\xi)\right)$ to the integrand yielding

$$f(x) = \frac{1}{\pi} \int_{-\infty}^\infty e^{-c^2\xi^2} e^{ix\xi}\, d\xi\,. \tag{3.37}$$

Using the FT, (3.33), to solve $f(x)$, (3.37):

$$\begin{aligned}
f(x) &= \frac{1}{\pi} \int_{-\infty}^\infty e^{-c^2\xi^2} e^{ix\xi}\, d\xi \\
&\quad \text{... set } \xi = -y \text{ ...} \\
&= -\frac{1}{\pi} \int_{\infty}^{-\infty} e^{-c^2(-y)^2} e^{ix(-y)}\, dy \\
&= \frac{1}{\pi} \left[\int_{-\infty}^\infty e^{-c^2y^2} e^{-iyx}\, dy \right] \\
&\quad \text{... note that the square bracket is the same as (3.33) where } b \to c \text{ and } \eta \to x \text{ ...} \\
&= \frac{1}{\pi} \left[\frac{\sqrt{\pi}}{|c|} e^{-x^2/(4c^2)} \right] \\
&= \frac{1}{|c|\sqrt{\pi}} e^{-x^2/(4c^2)}
\end{aligned}$$

Using the IFT, (3.34), to solve $f(x)$, (3.37):

$$\begin{aligned}
f(x) &= \frac{1}{\pi} \int_{-\infty}^\infty e^{-c^2\xi^2} e^{ix\xi}\, d\xi \\
&= \frac{2}{\sqrt{\pi}} \left[\frac{1}{2\pi} \int_{-\infty}^\infty \sqrt{\pi} e^{-c^2\xi^2} e^{ix\xi}\, d\xi \right] \\
&\quad \text{... set } c^2 = 1/(4b^2) \text{ ...}
\end{aligned}$$

$$= \frac{2\,|b|}{\sqrt{\pi}} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{\pi}}{|b|} e^{-\xi^2/(4b^2)} e^{ix\xi}\, d\xi \right]$$

... note that the square bracket is the same as (3.33) where $y \to x$ and $\eta \to \xi$ ...

$$= \frac{2\,|b|}{\sqrt{\pi}} \left[e^{-b^2 x^2} \right]$$

... back substitution by setting $b^2 = 1/(4c^2)$ and $|b| = 1/(2\,|c|)$ yields ...

$$= \frac{1}{|c|\,\sqrt{\pi}} e^{-x^2/(4c^2)}$$

EXAMPLE 3.5

Find the SinTF of $f(x) = xe^{-b^2x^2}$, where b is a fixed parameter.

Note that $f(x)$ is related to the first derivative of $g(x) = e^{-b^2x^2}$, because

$$f(x) = \frac{-1}{2b^2} g'(x)\,.$$

The SinTF of $f(x)$ is therefore given by

$$F_S(\xi) = \frac{i}{2} F(\xi)$$

$$= \left(\frac{i}{2}\right)\left(\frac{-1}{2b^2}\right) \mathcal{F}[g'(x)]_{x\to\xi}$$

... using the derivative rule for FTs ...

$$= \left(\frac{i}{2}\right)\left(\frac{-1}{2b^2}\right) (i\xi)^1 \mathcal{F}[g(x)]_{x\to\xi}$$

... this Fourier transform is known (see (3.32)) ...

$$= \frac{-i}{4b^2}(i\xi) \left[\frac{\sqrt{\pi}}{|b|} e^{-\xi^2/(4b^2)} \right]$$

$$= \frac{\sqrt{\pi}}{4\,|b|^3} \xi e^{-\xi^2/(4b^2)}.$$

PROBLEM SET 3: THE FOURIER TRANSFORM

1. **(a)** Prove the following properties of the Fourier transform:

(i) $\mathcal{F}\left[f(x-\alpha)\right]_{x\to\xi} = e^{-i\alpha\xi}F(\xi)$

(ii) $\mathcal{F}\left[f(\beta x)\right]_{x\to\xi} = \dfrac{1}{|\beta|}F(\xi/\beta)$

(iii) $\mathcal{F}\left[\dfrac{d^n}{dx^n}f(x)\right]_{x\to\xi} = (i\xi)^n F(\xi)$

(b) Formulate the relationship between the parity (even or odd) and "complexness" (real or imaginary) of $f(x)$ and $F(\xi)$. Then prove it. (You probably first need to *discover* it, then formulate it, then prove it.)

2. Without the help of a symbolic computation program, e.g. MATHEMATICA, find the Fourier transform of

(a)

$$f(x) = e^{-|x|}, \qquad x \in (-\infty, \infty)$$

(b)

$$f(x) = \begin{cases} x, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

(c)

$$f(x) = \begin{cases} \sin(\omega x), & |x| < a \\ 0, & |x| \geq a \end{cases}$$

Here a and ω are an arbitrary constants. Also give the integral representation of $f(x)$ by using the inverse Fourier transform. Simplify your expression (by hand) as far as possible.

Draw a simple graph of the integral representation in MATLAB (or any other suitable programming language) by using the rectangular rule for integration. Also show the function itself, $f(x)$, on the graph to show that it corresponds. (Include your code in your assignment.)

3. **(a)** Without the help of a symbolic computation program, e.g. MATHEMATICA, find the Fourier transform of

$$f(x) = e^{-a|x|}, \qquad a > 0\,.$$

Keep a constant (but arbitrary).

(b) Give the integral representation of $f(x)$ by using the inverse Fourier transform and draw a simple graph of the integral representation in MATLAB (or any other suitable programming language) by using the rectangular rule for integration. (Include your code in your assignment.)

(c) By using the result of **3(a)**, show that

$$\int_{-\infty}^{\infty} \left(\frac{\cos x}{1+x^2} \right) dx = \frac{\pi}{e} .$$

4. **(a)** Without the help of MATHEMATICA, find the Fourier transform of

$$f(x) = \begin{cases} \sin(n\pi x), & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

where n is an arbitrary constants.

(b) Give the integral representation of $f(x)$ by using the inverse Fourier transform. Simplify your expression (by hand) as far as possible.

Draw a simple graph of the integral representation in MATLAB (or any other suitable programming language), for n is not an integer, by using the rectangular rule for integration. Also show the function itself, $f(x)$, on the graph to show that it corresponds. (Include your code in your assignment.)

(c) Use the result of **4(b)** to find the definite integral of an expression on the interval $(-\infty, \infty)$. Choose your own expression.

5. **(a)** Without the help of a symbolic mathematical computation program, e.g. MATHEMATICA, find the Fourier transform of

$$f(x) = \begin{cases} 1 - |x| , & |x| < 1 \\ 0, & |x| \geq 1 . \end{cases}$$

(b) Give the integral representation of $f(x)$ by using the inverse Fourier transform. Simplify your expression (by hand) as far as possible.

Draw a simple graph of the integral representation in MATLAB (or any other suitable programming language) by using the rectangular rule for integration. Also show the function itself, $f(x)$, on the graph to show that it corresponds. (Include your code in your assignment.)

(c) By using the result of **5(a)**, find the Fourier transform, $G(\xi)$, of

$$g(x) = \frac{1 - \cos(px)}{x^2}$$

where p is a positive constant.

(d) Use the result of **5(b)** to find the definite integral of an expression on the interval $(-\infty, \infty)$. Choose your own expression.

6. First show that

$$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$$

and then show that

$$\mathcal{F}\left[e^{-(x/b)^2} \right]_{x \to \xi} = \sqrt{\pi} b e^{-\left(\frac{1}{2} b \xi\right)^2} \tag{1}$$

where b is a positive constant. Some hints on how to do this was covered in class.

Do not use the Fourier transform of the Gaussian function derived in Paragraph 3.5 as "a known fact".

7. Design your own function $f(x)$ defined on $x \in (-\infty, \infty)$ that contains an arbitrary parameter (say a). Then find the Fourier transform of $f(x)$ as well as its integral representation. Also write down the values of two definite integrals on $(-\infty, \infty)$. Use your own creativity.