3.4 The Fourier Transform

3.4.1 Formulation and examples

[*The formula for the Fourier coefficients links a periodic function $f(x)$ to an infinite set of coefficients c_k, and the formula for the Fourier series transforms these coefficients back to $f(x)$. There is therefore the notion of going 'back and forth' between $f(x)$ and c_k. We may consider the c_k to be a 'transform' of $f(x)$, and $f(x)$ to be the 'inverse transform' of c_k.*

The continuous Fourier transform extends this notion to functions $f(x)$ that are not periodic.]

We restate the Fourier formulae for a complex series here:

$$c(k) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik\pi x/L}dx \qquad \text{for } k \in \mathbb{Z}, \tag{3.1}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c(k)e^{ik\pi x/L}, \qquad \text{for } x \in \mathbb{R}. \tag{3.2}$$

Note that we have replaced c_k with a new notation $c(k)$, implying that c is a function of k — it is in fact a discrete function defined on $k \in \mathbb{Z}$.

It will be shown in the next section that the formulae for the Fourier Transform (FT) and the Inverse Fourier Transform (IFT), can be derived from (3.1) and (3.2).

If $f(x)$ is defined on $x \in (-\infty, \infty)$, and $f(x)$ is square integrable (i.e. $\int_{-\infty}^{\infty} |f(x)|^2 dx$ is finite) then the FT of $f(x)$ is given by $F(u)$ below. $F(u)$ is then also square integrable and $f(x)$ can be recovered form $F(u)$ by using the formula for the IFT.

Fourier Transform:

$$\boxed{F(u) = \int_{-\infty}^{\infty} f(x)e^{-ixu}dx} \tag{3.3}$$

Inverse Fourier Transform:

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{iux}du} \tag{3.4}$$

Note that f is a function of x, while F is a function of u. In practical applications x may be measured for example in m, then u is measured in m^{-1}, or x may be measured for example in s, then u is measured in s^{-1}, i.e. Hz. The variable x is referred to as 'physical space' while the variable u is referred to a 'frequency space'. Both $f(x)$ and $F(u)$ represent the same function: $f(x)$ is a physical space representation and $F(u)$ is a frequency space representation.

We shall now show some examples.

Example 3.4.1

Find the FT of $f(x)$, where

$$f(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| > 1. \end{cases}$$

Solution:

$$\begin{aligned} F(u) &= \int_{-\infty}^{\infty} f(x) e^{-iux} dx \\ &= \int_{-1}^{1} 1 \cdot e^{-iux} dx \\ &= \left[\frac{e^{-iux}}{-iu} \right]_{-1}^{1} \\ &= \frac{2}{u} \left[\frac{e^{-iu} - e^{iu}}{-2i} \right] \\ &= \frac{2\sin(u)}{u}. \end{aligned}$$

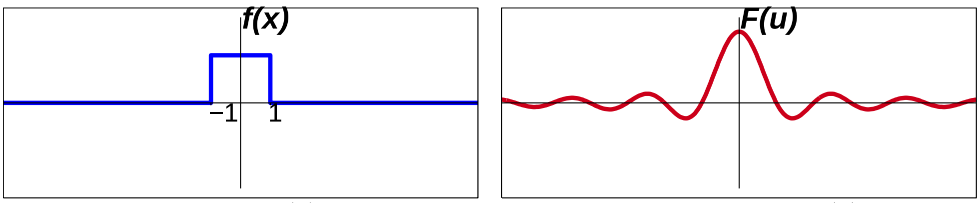

Figure 9. The function $f(x)$ of Example 3.4.1 and its Fourier Transform, $F(u)$.

Example 3.4.2

Find the FT of

$$f(x) = \begin{cases} \sin(\omega x) & |x| < 1, \\ 0 & |x| \geq 1. \end{cases}$$

where ω is a constant.

Solution:

$$
\begin{aligned}
F(u) &= \int_{-\infty}^{\infty} f(x)e^{-iux}dx \\
&= \int_{-1}^{1} \sin(\omega x)\cdot e^{-iux}dx \\
&= \int_{-1}^{1} \frac{e^{i\omega x}-e^{-i\omega x}}{2i}e^{-iux}dx \\
&= \frac{1}{2i}\left[\frac{e^{i(\omega-u)x}}{i(\omega-u)}-\frac{e^{i(-\omega-u)x}}{-i(\omega+u)}\right]_{1}^{1} \\
&= -i\left[\frac{e^{i(\omega-u)}-e^{-i(\omega-u)}}{2i(\omega-u)}+\frac{e^{-i(\omega+u)}-e^{i(\omega+u)}}{2i(\omega+u)}\right] \\
&= i\left[\frac{\sin(\omega+u)}{(\omega+u)}-\frac{\sin(\omega-u)}{(\omega-u)}\right].
\end{aligned}
$$

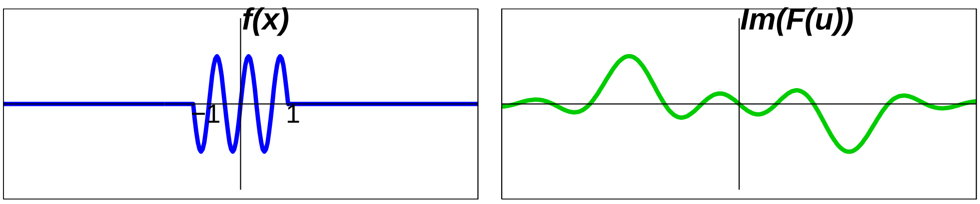

Figure 10. The function $f(x)$, of Example 3.4.2 and it Fourier Transform, $F(u)$.

Figure 11 shows the graphs of some square integrable functions together with their Fourier transforms. We have chosen functions that are all real and have definite parity. Their corresponding FT's then are either real or imaginary and also have definite parity. These properties will be discussed in a following section. The Fourier transform of a function with no particular parity, is also without parity.

We shall employ the notation that functions will be written with lowercase Latin symbols, and their corresponding FT's will be denoted by the corresponding uppercase letters. We shall further use the notation that the corresponding frequency of x is given by u and the corresponding frequency of y is given by v. For example, the FT of $g(x)$ is $G(u)$ and the FT of $r(y)$ is $R(v)$.

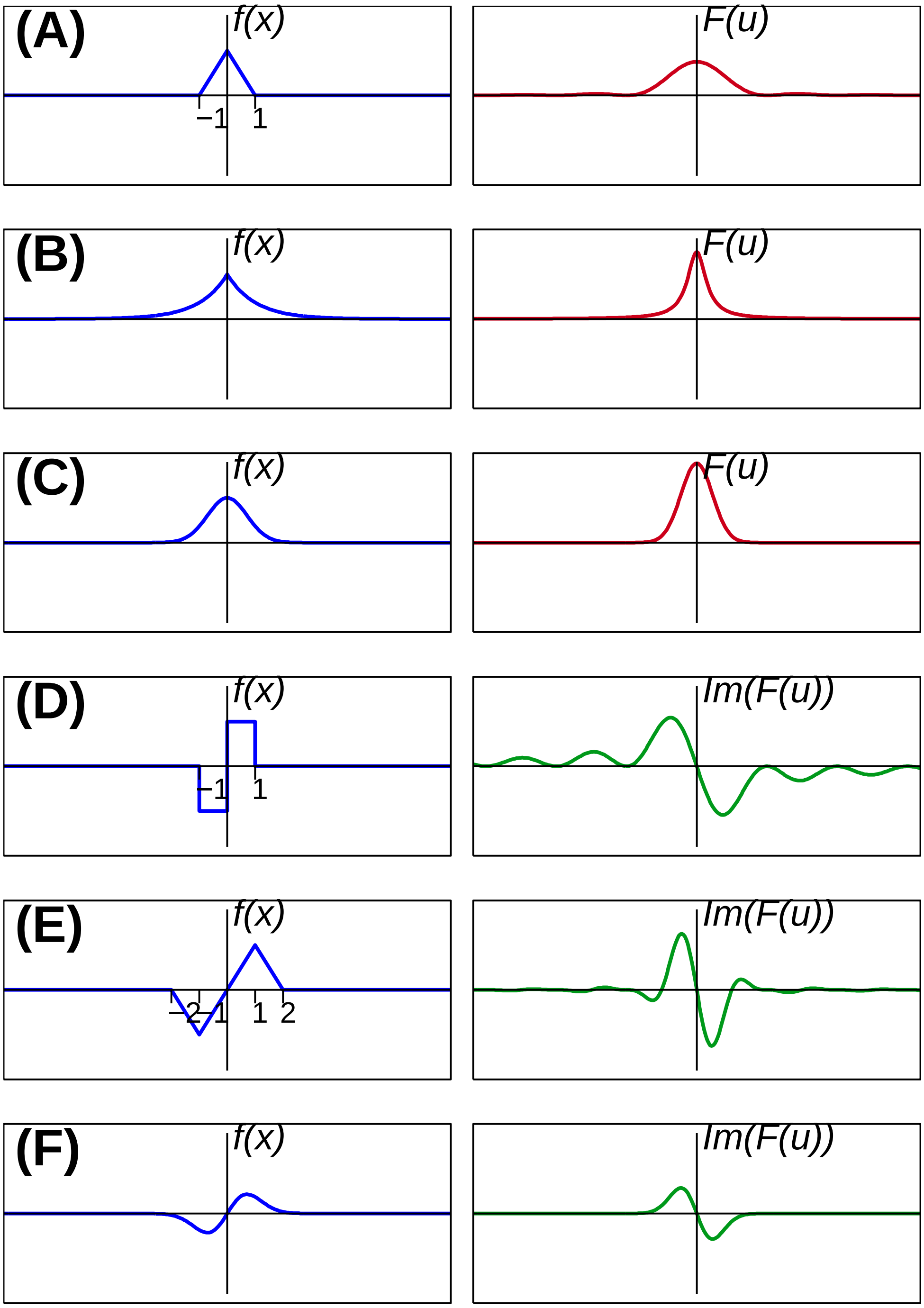

Figure 11. Some functions with their Fourier transforms.

	Function	Fourier Transform
(A)	$f(x) = \begin{cases} 1-\lvert x\rvert, & x \in [-1,1] \\ 0, & \text{otherwise} \end{cases}$	$F(u) = \frac{2(1-\cos u)}{u^2}$
(B)	$f(x) = e^{-\lvert x\rvert}$	$F(u) = \frac{2}{1+u^2}$
(C)	$f(x) = e^{-\frac{1}{2}x^2}$	$F(u) = \sqrt{2\pi}e^{-\frac{1}{2}u^2}$
(D)	$f(x) = \begin{cases} -1, & x \in [-1,0] \\ 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$	$F(u) = i\frac{2(\cos u-1)}{u}$
(E)	$f(x) = \begin{cases} -2-x, & x \in [-2,-1] \\ x, & x \in [-1,1] \\ 2-x, & x \in [1,2] \\ 0, & \text{otherwise} \end{cases}$	$F(u) = i\frac{4\sin u(\cos u-1)}{u^2}$
(F)	$f(x) = xe^{-x^2}$	$F(u) = -\frac{1}{2}i\sqrt{\pi}ue^{-u^2/4}$

3.4.2 Derivation of the FT from the Fourier series

[*The Fourier Transform is a continuous analogue of the semi discrete Fourier series. We shall now show how the Fourier series and the Fourier transform are related.*]

We shall start by assuming the formulae for the complex Fourier series:

$$c(k) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik\pi x/L}dx \qquad \text{for } k \in \mathbb{Z}, \tag{3.5}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c(k)e^{ik\pi x/L}, \qquad \text{for } x \in \mathbb{R}. \tag{3.6}$$

The Fourier series changes into the Fourier transform when L is increased to infinity, and the coefficients are scaled suitably.

A particular example

We shall take a particular example to study this effect. For $L > 1$ let

$$f(x) = \begin{cases} 0 & \text{if } x \in [-L,-1), \\ 1 & \text{if } x \in (-1,1), \\ 0 & \text{if } x \in (1,L]. \end{cases} \tag{3.7}$$

The complex Fourier coefficients of $f(x)$ are given by

$$c(k) = \frac{1}{2L}\int_{-1}^{1} 1 \cdot e^{-i\pi kx/L}dx = \frac{1}{\pi k}\sin(\pi k/L). \tag{3.8}$$

The coefficients may also be expressed as

$$c(k) = \frac{1}{L}\left(\frac{\sin(\pi k/L)}{\pi k/L}\right). \tag{3.9}$$

which may be recognised as the sinc function with argument $(\pi k/L)$, and the function is scaled by $1/L$.

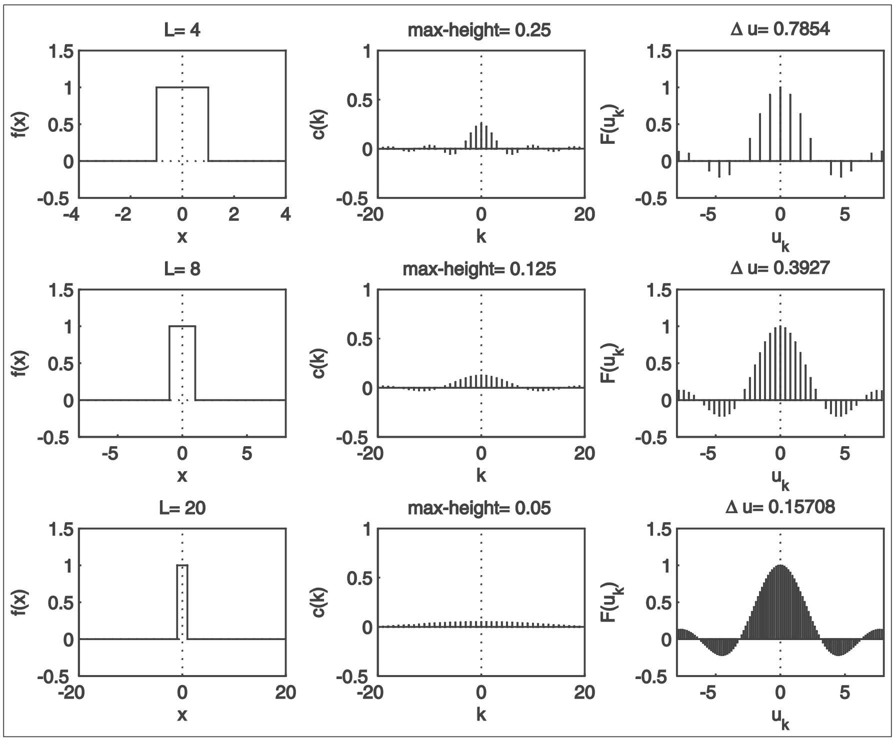

Figure 12. The rectangular pulse function, its FT and its scaled FT, for $L = 4, 8, 20$.

Figure 4.2 shows $f(x)$ in the first column and $c(k)$ in the second column for three values of L, viz. $L = 4$, $L = 8$ and $L = 20$. As would be expected from the $1/L$ scaling, the function $c(k)$ shrinks vertically with increasing L. Furthermore, $c(k)$ gets wider horizontally with increasing L, which is expected from the scaling of the argument by π/L.

In order to retain the basic function $\text{sinc}(\pi k/L)$, as $L \to \infty$, the shrinking and widening effect may be 'scaled back' as follows:

Let

$$u_k = \frac{\pi k}{L},$$

i.e.

$$u_k = k\Delta u, \quad k \in \mathbb{Z}, \qquad \text{where } \Delta u = \frac{\pi}{L},$$

and let

$$F(u_k) = 2Lc(k).$$

The reason for including a factor 2 is simply to end up later with a unit coefficient in stead of a $\frac{1}{2}$. For the particular example above,

$$F(u_k) = 2\left(\frac{\sin(u_k)}{u_k}\right).$$

Now as $L \to \infty$, $F(u_k)$ retains the basic shape of the sinc function. However, note that the increment Δu decreases to zero as L increases to infinity. This means that as $L \to \infty$, u_k becomes a continuous variable. The third column in Figure 4.2 shows $F(u_k)$ for the same three values of L. Note that the 'shape' of F stays constant but the spacing between consecutive values of u_k becomes smaller.

The general case

Look at the formulae for the complex Fourier series, again:

$$c(k) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik\pi x/L}dx \qquad \text{for } k \in \mathbb{Z}, \tag{3.10}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c(k)e^{ik\pi x/L}, \qquad \text{for } x \in \mathbb{R}. \tag{3.11}$$

Multiply equation (3.10) by L and replace $Lc(k)$ by $F(u_k)$,

$$F(u_k) = \int_{-L}^{L} f(x)e^{-ik\pi x/L}dx = \int_{-L}^{L} f(x)e^{-iu_k x}dx$$

Let $L \to \infty$ then u_k becomes continuous and

$$F(u) = \int_{-\infty}^{\infty} f(x)e^{-iux}dx. \tag{3.12}$$

This is the formula for the Fourier transform of $f(x)$ (compare to (3.3)).

How can $f(x)$ be recovered from $F(u)$? Take equation (3.11) and substitute u_k for $\pi k/L$, and also replace $c(k)$ by $F(u_k)/(2L)$,

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{1}{2L}F(u_k)e^{iu_k x}. \tag{3.13}$$

Note that $\frac{1}{2L} = \frac{\Delta u}{2\pi}$, so that (3.13) may be expressed as integration by using the rectangle rule,

$$f(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} F(u_k)e^{iu_k x}\Delta u. \tag{3.14}$$

Taking the limit as $L \to \infty$, yields

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{iux}du \tag{3.15}$$

which is the formula for the inverse Fourier transform of $F(u)$ (compare to (3.4)).

3.4.3 Integral representations

Since the IFT of the FT of a function $f(x)$, is equal to $f(x)$, we may express $f(x)$ as the IFT of the FT of $f(x)$. Let us try that with the function in Example 1.

$$f(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases},$$

and the FT of $f(x)$ is

$$F(u) = \frac{2\sin(u)}{u},$$

therefore $f(x)$ may also be expressed as

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(u)}{u} e^{ixu} du \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(u)}{u}\cos(xu)du + \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\sin(u)}{u}\sin(xu)du \end{aligned}$$

..... but the second integral is the integral of an odd function over a symmetric interval, and is therefore zero.

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(u)}{u}\cos(xu)du.$$

In other words:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(u)}{u}\cos(xu)du = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

This gives a new way to express the rectangular pulse function. It is called an *integral representation* of a piecewise defined function. In order to describe the rectangular pulse function mathematically, we may either use the piecewise definition (the right hand side), or define the function as a single expression containing an infinite integral (the left hand side).

3.4.4 The Gauss function

The function e^{-x^2} appears often in statistics and physics. It is called the Gauss function or just the *Gaussian* (and in statistics, a scaled version of it is called the normal distribution). Figure 13 shows this bell shaped function.

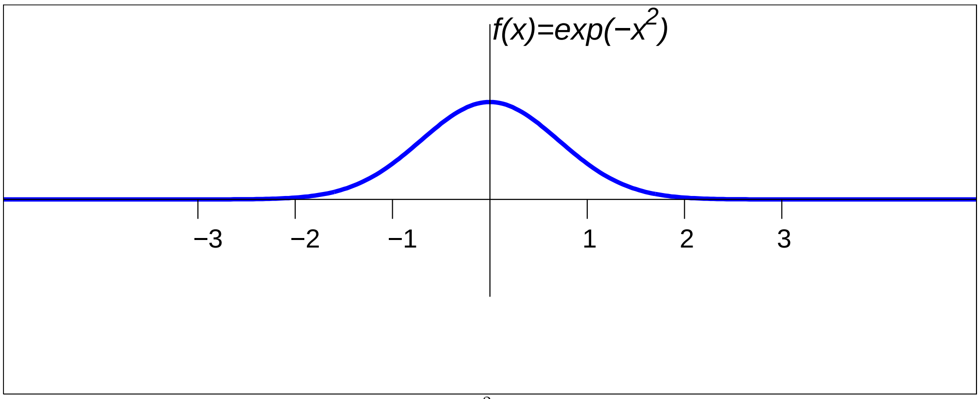

Figure 13. The Gauss function, $f(x) = e^{-x^2}$.

We shall now attempt to find the FT of this function. Let us first check the integral of the function i.e. $\int e^{-x^2}dx$. This integral is usually expressed in terms of a new special function called the *error-function*. However, if the limits of integration are infinite, the value of this integral can be found without the aid of the error function. The following special trick will be used.

Let

$$I = \int_{-\infty}^{\infty} e^{-x^2}dx,$$

then

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy. \end{aligned}$$

The right hand side is an area integral over the entire Cartesian plane. Now transform to polar coordinates: Let $r^2 = x^2 + y^2$, and $\tan\theta = y/x$, then the area element $dxdy$ is transformed to $rd\theta dr$ and the limits change to $r \in [0, \infty)$ and $\theta \in [0, 2\pi]$. Therefore

$$\begin{aligned} I^2 &= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2}rdrd\theta, \\ &= \int_{\theta=0}^{2\pi}\left[\frac{-e^{-r^2}}{2}\right]_0^{\infty} d\theta, \\ &= \int_0^{2\pi} \tfrac{1}{2}d\theta, \\ &= \pi, \end{aligned}$$

and then

$$I = \sqrt{\pi}.$$

In order to find the FT of the Gauss function, another special technique is used:

$$F(u) = \int_{-\infty}^{\infty} e^{-x^2}e^{-ixu}dx.$$

This integral cannot be found analytically, but let us use the following technique: Differentiate the function to u,

$$\begin{aligned}
\frac{\partial F(u)}{\partial u} &= \int_{-\infty}^{\infty} -ixe^{-x^2}e^{-ixu}dx, \\
&= \frac{i}{2}\left[e^{-x^2}e^{-ixu}\right]_{-\infty}^{\infty} - \frac{i}{2}\int_{-\infty}^{\infty}(-iu)e^{-x^2}e^{-ixu}dx, \\
&= -\frac{u}{2}F(u).
\end{aligned}$$

This is a differential equation in u,

$$\frac{\partial F(u)}{\partial u} = -\frac{u}{2}F(u).$$

The solution is:

$$F(u) = Ke^{-u^2/4},$$

where K is an integration constant. However note that

$$F(0) = \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi},$$

therefore $K = \sqrt{\pi}$ and

$$F(u) = \sqrt{\pi}e^{-u^2/4}.$$

This is an interesting FT pair: The Gauss function is (apart from some scaling) its own Fourier transform !

3.4.5 Properties of the FT

[*Why would anyone want to compute the Fourier Transform of a function ? It is because some of the properties of the FT allow us to do some operations much easier in the frequency domain than in the physical domain. After the operation has been completed we can return to physical space by applying the inverse FT.*

We shall now list many of the properties of the FT (most without derivation).
]

<u>Notation:</u> We shall use the notation

$$\mathcal{F}\left[f(x)\right]_{x\to u} = F(u),$$

which means *the Fourier transform of f is F and the physical variable x is associated with the frequency variable u.*

The inverse FT will be denoted by

$$\mathcal{F}^{-1}\left[F(u)\right]_{u\to x} = f(x).$$

which means *the Inverse Fourier transform of F is f and the frequency variable u is associated with the physical variable x.*

Linearity:

$$\mathcal{F}\left[\alpha f(x) + \beta g(x)\right]_{x\to u} = \alpha F(u) + \beta G(u).$$

This is easily proven by using the linearity of integration.

Parity and complex values:

It can be shown that the following is true of functions with definite parity and which is real only or imaginary only:

$f(x)$	$F(u)$
real & even	real & even
real & odd	imaginary & odd
imaginary & even	imaginary & even
imaginary & odd	real & odd

Attempt to formulate the results of this table in a more concise way.

Scaling the argument:

$$\mathcal{F}\left[f(\beta x)\right]_{x\to u} = \frac{1}{|\beta|} F(u/\beta)$$

Physically this means that when a function is stretched wider, its Fourier transform becomes narrower and vice versa. It is this effect that is responsible for the *Heisenberg uncertainty principle* in physics. This is because the momentum wave function of a particle is the FT of the position wave function.

Shifting the argument:

$$\mathcal{F}\left[f(x+\omega)\right]_{x\to u} = e^{i\omega u} F(u)$$

This effect is often used in applications to classify functions regardless of their particular shift.

Differentiation:

$$\mathcal{F}\left[\frac{d^n f(x)}{dx^n}\right]_{x\to u} = (iu)^n F(u)$$

This is perhaps one of the most useful properties of the Fourier transform. It means that differentiating n times in physical space is equivalent to multiplying the FT by the n-th power of iu in frequency space.

This formula relates the n-th power of a function to its n-th derivative. For square integrable functions, this formulation provides a way to compute *fractional derivatives* of a function.

This property will be derived here only for the case $n = 1$. You may use induction to generalize this result for all positive integer n.

You are reminded that square integrable functions have the property that $\lim_{x\to\pm\infty} f(x) = 0$. This is also true for any derivative of a square integrable function.

$$
\begin{aligned}
\mathcal{F}\left[f'(x)\right]_{x\to u} &= \int_{-\infty}^{\infty} f'(x)e^{-ixu}dx \\
&= \left[f(x)e^{-ixu}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\frac{d}{dx}e^{-ixu}dx \\
&\text{.... the first term is zero because } f(x) \text{ is square integrable} \\
&= -(-iu)\int_{-\infty}^{\infty} f(x)\frac{d}{dx}e^{-ixu}dx \\
&= iuF(u)
\end{aligned}
$$

3.5 The SIN and COS transforms

[*The Fourier transform is applied to functions defined on* $x \in (-\infty, \infty)$, *and therefore it can only be used in the solution od pdes where the domain is* $x \in (-\infty, \infty)$. *Often the domain is rather* $x \in [0, \infty)$, *and we need to have a transform with appropriate derivative properties that acts on this domain. Both the Sin Transform and the Cos transform can perform this role.*

With both the Sin transform and the Cos transform, there is, however, an extra piece of information that is needed at the point $x = 0$ *in order to apply their respective derivative properties. But, in a well-posed boundary condition on the domain* $x \in [0, \infty)$, *there is also an extra condition needed at the point* $x = 0$, *and this supplies the information needed by the Sin or Cos transform.*
]

3.5.1 The Cos Transform

We now introduce a new transform called the *Cos-transform*, abbreviated with CosTF. It is actually not really new, it is just a variant of the FT.

We shall use the notation

$$\mathcal{C}[f(x)]_{x\to\xi},$$

to denote the CosTF of the function $f(x)$, where the frequency variable associated with x is ξ. We shall also often employ the shorter notation $F_C(\xi)$ to denote the CosTF of $f(x)$.

Consider an <u>even real</u> function $f(x)$ defined on $x \in (-\infty, \infty)$. Its Fourier transform is given by

$$\begin{aligned}
F(\xi) &= \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx \\
&= \int_{-\infty}^{\infty} f(x)\cos(x\xi)dx - i\int_{-\infty}^{\infty} f(x)\sin(x\xi)dx
\end{aligned}$$

..... The latter integral disappears because $f(x)\sin(x\xi)$ is odd.

$$= 2\int_{0}^{\infty} f(x)\cos(x\xi)dx.$$

Let us define the relationship between the CosTF and the FT as

$$F_C(\xi) = \frac{1}{2}F(\xi), \tag{3.16}$$

In other words

$$\boxed{\mathcal{C}[f(x)]_{x\to\xi} = \int_0^{\infty} f(x)\cos(x\xi)dx.} \tag{3.17}$$

Although the original $f(x)$ was defined over the entire real line, the CosTF can be viewed as a transform for functions only defined on $x \in [0, \infty)$. Note that $F_C(\xi)$ is an even function of ξ, and at this stage it may be defined for all $\xi \in (-\infty, \infty)$, even though x lies only in $[0, \infty)$.

In order to transform back, note that

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2F_C(\xi)e^{ix\xi}d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2F_C(\xi)\cos(x\xi)dx + i\int_{-\infty}^{\infty} 2F_C(\xi)\sin(x\xi)dx
\end{aligned}$$

..... Once again, the latter integral disappears because $F_C(\xi)\sin(x\xi)$ is odd

$$= \frac{2}{\pi}\int_0^{\infty} F_C(\xi)\cos(x\xi)dx$$

Therefore *Inverse Cos transform*, abbreviated by ICosTF, is given by

$$\boxed{\mathcal{C}^{-1}[F_C(\xi)]_{\xi\to x} = \frac{2}{\pi}\int_0^{\infty} F_C(\xi)\cos(x\xi)d\xi.} \tag{3.18}$$

We now have a transform that operates on the region $[0, \infty)$, and this can be used to solve boundary value problems on the semi-infinite domain $x \in [0, \infty)$, $t \in [0, \infty)$.

3.5.2 The Sin Transform

There is a similar transform that operates on the domain $x \in [0, \infty)$. It is called the *Sin-transform*, and is abbreviated by SinTF. We shall likewise also use the $\mathcal{S}[f(x)]_{x \to \xi}$ to denote the SinTF of a function $f(x)$, and Similarly, we shall also use the shorter notation $F_S(\xi)$ to denote the SinTF of a function $f(x)$.

The SinTF can be derived from the FT by considering an *odd real* function $g(x)$ defined on $x \in (-\infty, \infty)$. The Fourier transform of $g(x)$ is given by

$$\begin{aligned} G(\xi) &= \int_{-\infty}^{\infty} g(x) e^{-ix\xi} dx \\ &= \int_{-\infty}^{\infty} g(x) \cos(x\xi) dx - i \int_{-\infty}^{\infty} g(x) \sin(x\xi) dx \\ &\quad \text{..... The first integral disappears because } g(x)\cos(x\xi) \text{ is odd} \\ &= -2i \int_{0}^{\infty} g(x) \sin(x\xi) dx \end{aligned}$$

We now define the relationship between the SinTF and the FT as

$$\mathcal{F}_S[f(x)]_{x \to \xi} = \frac{i}{2} G(\xi). \tag{3.19}$$

In other words

$$\boxed{G_S(\xi) = \int_{0}^{\infty} g(x) \sin(x\xi) dx.} \tag{3.20}$$

Note that $G_S(\xi)$ is an odd function of ξ, because it is the imaginary part of the FT of a real odd function.

In order to transform back, note that

$$\begin{aligned} g(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{i} G_S(\xi) e^{ix\xi} d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} -2i G_S(\xi) \cos(x\xi) dx + i \int_{-\infty}^{\infty} 2i G_S(\xi) \sin(x\xi) dx \\ &\quad \text{..... Once again, the first integral disappears because } G_S(\xi)\cos(x\xi) \text{ is odd.} \\ &= \frac{2}{\pi} \int_{0}^{\infty} G_S(\xi) \sin(x\xi) dx. \end{aligned}$$

Therefore *Inverse Sin-transform*, abbreviated by ISinTF, is given by

$$\boxed{\mathcal{S}^{-1}[F_S(\xi)]_{\xi \to x} = \frac{2}{\pi} \int_{0}^{\infty} F_C(\xi) \sin(x\xi) d\xi.} \tag{3.21}$$

Both the CosTF and the SinTF are defined only for square integrable functions on the interval $[0, \infty)$.

3.5.3 The SinTF and CosTF of some functions

It is often easier to find the FT of a function defined on $[0, \infty)$, but extended to $(-\infty, \infty)$ as even (in the case of the CosTF) or odd (in the case of the SinTF) and then to use (3.16) or (3.19) to find its appropriate Sin or Cos transform. We shall do a few examples below.

Example 1:

Find the CosTF of $f(x) = e^{-b^2x^2}$, where b is a fixed parameter.

The FT of $f(x)$ is

$$F(\xi) = \frac{\sqrt{\pi}}{b} e^{-\xi^2/(4b^2)}$$

Using (3.16), then

$$F_C(\xi) = \frac{1}{2} F(\xi) = \frac{\sqrt{\pi}}{2b} e^{-\xi^2/(4b^2)}$$

Example 2:

Find the ICosTF of

$$F_C(\xi) = e^{-c^2\xi^2}, \quad x \in [0, \infty)$$

where c is a fixed parameter. Once again we shall make use of what we already know of the Fourier transform to help us.

The FT of $f(x)$ is

$$F(\xi) = \frac{\sqrt{\pi}}{c} e^{-\xi^2/(4c^2)}$$

We want to find

$$f(x) = \frac{2}{\pi} \int_0^\infty e^{-c^2\xi^2} e^{ix\xi} d\xi.$$

Extend $e^{-c^2\xi^2}$ so that it forms an even function on $\xi \in (-\infty, \infty)$. Actually in this form it is already even on $\xi \in (-\infty, \infty)$.

Then

$$
\begin{aligned}
f(x) &= \frac{2}{\pi}\int_0^\infty e^{-c^2\xi^2} e^{ix\xi} d\xi \\
&= \frac{1}{\pi}\int_{-\infty}^\infty e^{-c^2\xi^2} e^{ix\xi} d\xi \\
&= \frac{2k}{\sqrt{\pi}}\left[\frac{1}{2\pi}\int_{-\infty}^\infty \frac{\sqrt{\pi}}{k} e^{\xi^2/(4k^2)} e^{ix\xi} d\xi\right]
\end{aligned}
$$

.... where $c^2 = 1/(4k^2)$, and note how we have deliberately shaped the integrand, so that it is a known FT.

$$
\begin{aligned}
&= \frac{2k}{\sqrt{\pi}} e^{-k^2x^2} \\
&= \frac{1}{c\sqrt{\pi}} e^{-x^2/(4c^2)}, \qquad \text{... after substituting } c = 1/(2k) \text{ back.}
\end{aligned}
$$

Example 3:

Find the SinTF of $f(x) = xe^{-b^2x^2}$, where b is a fixed parameter.

Note that $f(x)$ is related to the first derivative of $g(x) = e^{-b^2x^2}$, because

$$
f(x) = \frac{-1}{2b^2} g'(x)
$$

therefore the SinTF of $f(x)$ is given by

$$
\begin{aligned}
F_S(\xi) &= \frac{i}{2} F(\xi) \\
&= \left(\frac{i}{2}\right)\left(\frac{-1}{2b^2}\right)\mathcal{F}[g'(x)]_{x\to\xi} \\
&= \left(\frac{i}{2}\right)\left(\frac{-1}{2b^2}\right)(i\xi)^1\mathcal{F}[g(x)]_{x\to\xi}
\end{aligned}
$$

..... using the derivative rule for FT's.

$$
\begin{aligned}
&= \frac{-i}{4b}(i\xi)\left[\frac{\sqrt{\pi}}{b} e^{-\xi^2/(4b^2)}\right] \\
&= \frac{\sqrt{\pi}}{4b^3}\xi e^{-\xi^2/(4b^2)}.
\end{aligned}
$$

3.5.4 Derivative properties of the Cos and Sin transforms

For our purpose it is the second derivative that is used in pdes. We shall therefore only consider the appropriate transform of the second derivative of a function here.

Consider

$$
\begin{aligned}
\mathcal{C}[f''(x)]_{x\to\xi} &= \int_0^\infty f''(x)\cos(x\xi)dx \\
&= \Big[f'(x)\cos(x\xi)\Big]_0^\infty - \int_0^\infty f'(x)(-\xi)\sin(x\xi)dx \\
&= [0 - f'(0)] + \xi\int_0^\infty f'(x)\sin(x\xi)dx \\
&= -f'(0) + \xi\left(\Big[f(x)\sin(x\xi)\Big]_0^\infty - \int_0^\infty f(x)(\xi)\cos(x\xi)dx\right) \\
&= -f'(0) + \xi\Big(0 - f(0)\times 0 - \xi F_C(\xi)\Big) \\
&= -f'(0) - \xi^2 F_C(\xi)
\end{aligned}
$$

Therefore

$$\boxed{\mathcal{C}[f''(x)]_{x\to\xi} = -f'(0) - \xi^2 F_C(\xi)} \tag{3.22}$$

There is a similar identity for the SinTF of a second derivative, and it is derived in similar manner. We shall not derive it here, but simply state it as,

$$\boxed{\mathcal{S}[f''(x)]_{x\to\xi} = \xi f(0) - \xi^2 F_S(\xi)} \tag{3.23}$$