Chapter 3

Fourier Analysis

Fourier analysis is concerned with the decomposition of periodic functions into sine and cosine components. A series of sine and cosine functions often form part of a general solution of a pde. When an arbitrary function is supplied as a boundary for a value problem, one has to determine how this function can be represented as an infinite series of sine and/or cosine terms. Fourier analysis provides the basis for performing this task.

Fourier analysis has various generalizations, and it may be said that in general Fourier Analysis is a method to transform from the physical space to frequency space and back. There are three types of Fourier transformations available. They are listed in Table 1.

Table 1

THE FOURIER SERIES	(FS)	transforms between a periodic function and an infinite discrete series of coefficients
THE FOURIER TRANSFORM (also called the CONTINUOUS FOURIER TRANSFORM)	(FT) (CFT)	transforms between two continuous functions defined on $(-\infty, \infty)$
THE DISCRETE FOURIER TRANSFORM	(DFT)	transforms between two discrete periodic series

In these notes only the FS and the FT will be discussed, as only these are used in the solution of boundary value problems.

3.1 The Fourier Series

3.1.1 Periodicity

[*Since Fourier analysis has to do with periodic functions, a section on the definition and properties of periodic functions is given.*]

Periodic functions such as $\sin x$ and $\cos x$ are well known. These functions repeat the same pattern on consecutive intervals with width 2π. The interval width for repetition is called the *period*.

A function $p(x)$ is *periodic* with *period* K if

$$p(x+K) = p(x) \qquad \text{for all real } x.$$

The terminology "$p(x)$ is K*-periodic*" is often also employed.

Note that if a function is K-periodic, then it is automatically also $2K$-periodic, $3K$-periodic, etc. The period is, however, defined as the smallest interval width over which the function is periodic.

Three examples of periodic functions that occur often, are shown in Figure 1, and listed in Table 2.

Table 2

SQUARE WAVE FUNCTION $p_{cont.}(x) = \begin{cases} 1, & x \in (2k, 2k+1) \\ -1, & x \in (2k-1, 2k) \end{cases}$ for $k \in \mathbb{Z}$
ROOF-TOP FUNCTION $p_{cont.}(x) = \begin{cases} x-2k, & x \in [2k, 2k+1) \\ -x+2k, & x \in [2k-1, k) \end{cases}$ for $k \in \mathbb{Z}$
SAW-TOOTH FUNCTION $p_{cont.}(x) = x - 2k, \quad x \in (2k-1, 2k+1)$ for $k \in \mathbb{Z}$

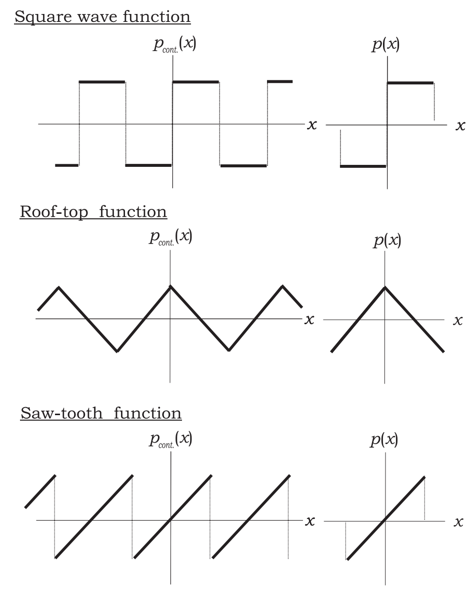

Figure 1. Some well-known periodic functions.

If the periodic function is known over one period, then it is known for all real x, and therefore a periodic function is often given by simply supplying its value over only one period or 'window'. For example, the square wave may be given as

$$p(x) = \begin{cases} 1, & x \in (0,1), \\ -1, & x \in (-1,0). \end{cases}$$

The periodic function for the square wave in the table above, is called the *periodic continuation* of $p(x)$, and we shall denote it by $p_{cont.}(x)$.

3.1.2 Parity

[*Parity has to do with 'oddness' or 'evenness' of a function. Knowing the parity of a function and using its properties, helps to reduce work when the Fourier coefficients are calculated.*]

A function whose graph is a mirror image about the y-axis is an *even* function. A function whose graph consists of an upside-down image on the other side of the y-axis, is an *odd* function. Some examples are shown in Figure 2. By definition

A function $s(x)$ is *even* if $s(-x) = s(x)$ for all real x.
A function $a(x)$ is *odd* if $a(-x) = -a(x)$ for all real x.

Examples of even functions are x^2, x^4, e^{-x^2} and $\cos x$. Examples of odd functions are x, x^3, $1/x$ and $\sin x$. A properly defined odd function $a(x)$ must have the property that $a(0) = 0$.

Reference to whether a function is even or odd, is collectively called the *parity* of the function. It is said that a function that is neither even nor odd has 'no parity'.

The term 'parity' is also applied to integers. For example, if a set of functions $\{\phi_0, \phi_1, ...\}$ is given and ϕ_0,ϕ_2,ϕ_4,... are even, while ϕ_1, ϕ_3, ... are odd, it is said that the functions and their indices have 'the same parity'.

Parity decomposition: Any function $f(x)$ may be decomposed into a even part and an odd part. If

$$f(x) = s(x) + a(x),$$

then the even component is

$$s(x) = \tfrac{1}{2}(f(x) + f(-x)),$$

and the odd component is

$$a(x) = \tfrac{1}{2}(f(x) - f(-x)).$$

Products:

Products of functions with parity are as follows:

even × even = even

even × odd = odd

odd × odd = even

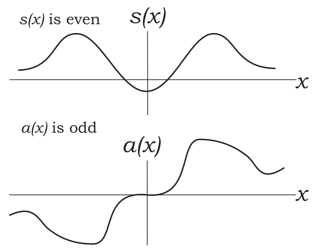

Figure 2. Examples of an even function and an odd function.

Derivatives:

All the even derivatives of a function have the same parity as the function, while the odd derivatives have opposite parity to the function.

Integrals over a symmetric interval:

Integrals over $[-K, K]$ simplify when the integrand has some parity. Let $s(x)$ be even and $a(x)$ be odd, then

$$\int_{-K}^{K} a(x)dx = 0,$$

and

$$\int_{-K}^{K} s(x)dx = 2\int_{0}^{K} s(x)dx.$$

3.1.3 Fourier Series

[*This section shows how a Fourier series is obtained, without derivation of the formulae. It also illustrates how a truncated Fourier series approximates the function from which it was calculated. The derivation of the formulae for the Fourier coefficients will be done in the next section.*]

A Fourier series is an (infinite) series of sine and cosine functions that approximates a given function $f(x)$ over the interval $[-L, L]$. A *truncated* Fourier series is a finite series consisting of the first N terms of the infinite Fourier series. Such a truncated series is often called a *Fourier approximation.* Denote the truncated Fourier series of $f(x)$ with N terms, by $f_N(x)$ and the infinite Fourier series by $f_\infty(x)$, then

$$\boxed{f_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(k\pi x/L) + \sum_{k=1}^{N} b_k \sin(k\pi x/L)}, \qquad (3.1)$$

where

$$
\boxed{\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\
a_k &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(k\pi x/L)dx \qquad k = 1, 2, ..., N \\
b_k &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(k\pi x/L)dx \qquad k = 1, 2, ..., N
\end{aligned}}
\tag{3.2}
$$

The constants $a_k, \quad k = 0, 1, ...$ and $b_k, \quad k = 1, 2, ...,$ are called the *Fourier coefficients*.

The functions used in the expansion, i.e. $\sin(k\pi x/L)$ and $\cos(k\pi x/L)$ are called *basis functions*. The basis functions in this case form an *orthogonal set*. More about sets of orthogonal functions will be given in the next section.

The index k is often called the *frequency* because for larger k, more oscillations are seen in the relevant basis function. The factor $\frac{k\pi}{L}$ corresponds to the physical concept of *angular frequency* of an oscillation.

The value of each coefficient is an indication of "how much" of the basis function associated with it, is "present" in the function $f(x)$. For example, if $f(x)$ is an even function, then only even basis functions (i.e. the cosines) will be present in the function and all the b_k's will be zero. If $f(x)$ displays wild oscillations then those Fourier coefficients with larger k will be relatively greater than for a function that does not show oscillations.

Example 1: Find the Fourier series of

$$
f(x) = \begin{cases} -1, & x \in (-\pi, 0), \\ \;\;1, & x \in (0, \pi). \end{cases}
$$

Since $f(x)$ is odd, $a_k = 0$, for all $k = 0, 1, ...$

$$
\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\sin(k\xi)d\xi \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -\sin(k\xi)d\xi + \frac{1}{\pi}\int_{0}^{\pi} \sin(k\xi)d\xi \\
&= \frac{1}{k\pi}\Big[\cos(k\xi)\Big]_{-\pi}^{0} + \frac{1}{k\pi}\Big[-\cos(k\xi)\Big]_{0}^{\pi} \\
&= \frac{1}{k\pi}\Big[1-(-1)^k\Big] + \frac{1}{k\pi}\Big[-(-1)^k-(-1)\Big] \\
&= \begin{cases} \dfrac{4}{k\pi} & \text{for } k \text{ odd}, \\[2ex] 0 & \text{for } k \text{ even}. \end{cases}
\end{aligned}
$$

Therefore

$$f_\infty(x) = \frac{4}{\pi}\left[\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + ...\right].$$

Odd numbers may be expressed as $(2j-1)$ for $j = 1, 2, ...$ and therefore the Fourier series may be expressed as follows,

$$f_N(x) = \frac{4}{\pi}\sum_{j=1}^{[(N+1)/2]} \frac{\sin((2j-1)x)}{2j-1}, \quad x \in [-L, L].$$

Actually $f_N(x)$, for $x \in \mathbb{R}$ is an approximation of the periodic continuation of $f(x)$, i.e. of the following function,

$$f_{cont.}(x) = \begin{cases} -1 & x \in ((2k-1)\pi, 2k\pi), \\ \ \ 1 & x \in (2k\pi, (2k+1)\pi), \end{cases} \qquad k \in \mathbb{Z}, \ \ x \in \mathbb{R}.$$

Figure 3 shows $f_{continued}(x)$ as well as $f_N(x)$ for $N = 1, 3, 5, 11$ and 23. Notice how the approximation improves with increasing N. Also notice how the Fourier approximation finds it difficult to approximate well near the discontinuities in $f_{continued}(x)$. This behaviour (overshoot and strong oscillations close to the discontinuity) is called the *Gibbs phenomenon*. If the periodic continuation of the function $f(x)$ does not contain any discontinuities, then the Gibbs phenomenon is absent. The next example illustrates this.

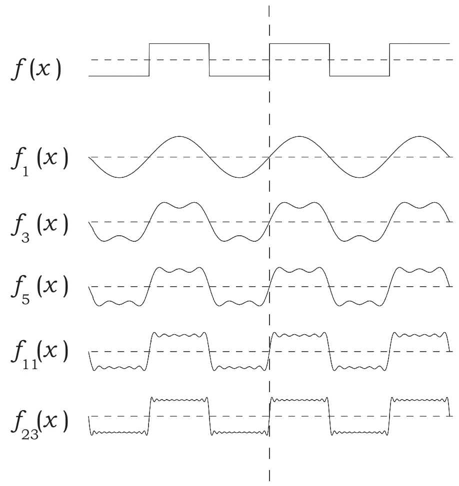

Figure 3. Fourier approximations of $f(x)$.

<u>Example 2:</u> Find the Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$.

Since $f(x)$ is even, all $b_k = 0$.

$$a_0 = \frac{2}{\pi}\int_0^\pi \xi d\xi = \frac{1}{\pi}\left[\frac{\pi^2}{2}\right] = \pi.$$

$$a_k = \frac{2}{\pi}\int_0^\pi \xi\cos(k\xi)d\xi = \frac{2}{k^2\pi}[(-1)^k - 1] = \begin{cases} \dfrac{-4}{k^2\pi}, & \text{for } k \text{ odd,} \\ 0, & \text{for } k \text{ even.} \end{cases}$$

Therefore

$$f_\infty(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + ...\right],$$

or in terms of the summation symbol

$$f_N(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=1}^{[(N+1)/2]}\left(\frac{\cos((2j-1)x)}{(2j-1)^2}\right).$$

Figure 4 shows a periodic continuation of $f(x)$ as well as $f_N(x)$ for $N = 1, 3, 5, 7$ and 15. Notice that the Gibbs phenomenon is absent.

<u>*Convergence:*</u>
We shall not discuss the convergence of Fourier series in detail, but only mention the following:
If $f(x)$ is continuous at x_0, then

$$\lim_{N\to\infty} f_N(x_0) = f(x_0) \qquad (3.3)$$

If $f(x)$ is discontinuous at x_0, then

$$\lim_{N\to\infty} f_N(x) = \frac{1}{2}\Big[f(x_0 - 0) + f(x_0 + 0)\Big] \qquad (3.4)$$

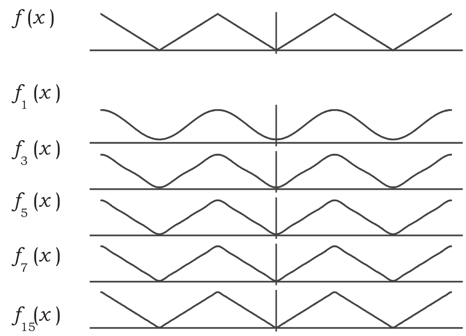

Figure 4. Some Fourier approximations of $f(x)$.

For example, the Fourier series of Example 1 is discontinuous at $x_0 = 0$, and therefore the average over the discontinuity is $\frac{1}{2}(1-1) = 0$. This may be confirmed by calculating $f_\infty(0)$:

$$f_\infty(0) = \frac{4}{\pi}\left[\sin(0) + \frac{\sin(0)}{3} + \frac{\sin(0)}{5} + ...\right] = 0$$

3.2 Orthogonal functions and Approximation

3.2.1 Function norms and the inner product

[*The Fourier approximation formulae are derived from the fact that the sine and cosine functions in the series form a complete orthogonal set.*

In order to understand the orthogonality concept, the idea of the "magnitude of a function" (the function norm) and "how close two functions are to each other" (the inner product), must first be introduced.]

It may be helpful utilize the analogy between vector norms and function norms, and likewise the analogy between vector dot products and the function inner product in order to see where the definitions of function norms and function inner products come from. In a more general setting, these concepts are actually considered entirely equivalent.

Let us recall the idea of vector dot products and vector norms.

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then the *dot product* of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a}^T\mathbf{b} = \sum_{j=1}^{n} a_j \bar{b}_j,$$

and the (Euclidean) *norm* of $\mathbf{a}$ is

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}^T\mathbf{a}} = \sqrt{\sum_{j=1}^{n} a_j \overline{a_j}},$$

where the overbar denotes the complex conjugate.

For functions $f(x)$ and $g(x) : [a, b] \to \mathbb{C}$ (i.e. the function values may be complex but x is real and on $[a, b]$), we define

The *inner product* of $f(x)$ and $g(x)$ on $[a, b]$ is

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx \tag{3.5}$$

and

The *norm* of $f(x)$ on the interval $[a, b]$ is

$$\|f(x)\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b f(x)\overline{f(x)}dx} \tag{3.6}$$

where, again, the overbar denotes the complex conjugate.

Note how summation in the vector sense is replaced by integration in the function sense.

A function is *normalized* when it is scaled such that its norm is 1.

Example: normalize $f(x) = x^2$ on the interval $[-1, 1]$.

Solution: $f(x)_{normalized} = \sqrt{\frac{5}{2}}x^2$.

The physical interpretation of the dot and inner products may be useful: Two normalized vectors point in approximately the same direction when their dot product is close to one. Likewise, two normalized functions are "close to each other" when their inner product is close to one.

When the dot product of two vectors is zero, they make a 90° angle w.r.t. each other, and the projection of one vector on the other is zero. One could say that when the dot product of two vectors is zero, then one vector ‘contains nothing‘ of the other vector. Similarly, two functions ‘contain nothing of each other’ if their inner product is zero. An example is $\langle \sin(x), x^2 \rangle = 0$ on any symmetric interval. The Taylor series of $\sin x$, i.e. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$, illustrates this — there is no x^2 in the series.

3.2.2 Orthogonality

Orthogonal functions:

Two functions are orthogonal over a given interval if their inner product over that interval is zero.

$f(x)$ and $g(x)$ is *orthogonal on* $[a, b]$ if

$$\int_a^b f(x)\overline{g(x)}dx = 0. \tag{3.7}$$

Example 2: Show that $f(x) = x - \frac{3}{4}$ and $g(x) = x^2$ are orthogonal on $[0, 1]$.

Example 3: Show that $f(x) = \sin x$ and $g(x) = \cos x$ are orthogonal on $[-\pi, \pi]$.

Orthonormal functions:

’n Set of orthonormal vectors form a convenient basis for a vector space and it simplifies the obtaining of coefficients for the vector representation in this basis. Likewise a set of orthonormal functions on $[a, b]$, will simplify the finding of coefficients of a function representation in the basis spanned by the orthogonal set of functions.

The set of functions $\{\phi_j(x),\ j = 0, 1, ...N\}$ are *orthonormal on* $[a, b]$ if

$$\int_a^b \phi_j(x)\overline{\phi_k(x)}dx = \begin{cases} 0, & \text{when} \quad j \neq k, \\ 1, & \text{when} \quad j = k. \end{cases} \tag{3.8}$$

Example 4: Show that the following set of functions is an orthonormal set on the interval $[-1, 2]$:

$$\begin{aligned} \phi_0(x) &= \frac{1}{\sqrt{3}}, \\ \phi_1(x) &= \frac{1}{3}(2x - 1), \\ \phi_2(x) &= \sqrt{\frac{20}{27}}(x^2 - x - \frac{1}{2}). \end{aligned}$$

3.2.3 Approximations in general

[*Approximating one function by another does not necessarily need the concept of orthogonality. In this section we shall discuss approximation techniques in general without referring to orthogonality.*]

We first consider the general problem of approximation a given function $f(x)$ on a given interval $[a, b]$ by another function $p(x)$ that contains a number of parameters. These parameters must be chosen in such a way that the approximation is 'as good as possible' in some sense.

Suppose $p(x)$ is a linear combination of a set of basis functions, while $f(x)$ may or may not lie in the function space spanned by these basis functions. We shall now discuss the so called *least-squares-approximation* idea.

The least-squares-approximation:

The least-squares-approximation finds $p(x)$ such that the function norm of the difference between $f(x)$ and $p(x)$ over the given interval is as small as possible.

Let

$$E = \|f(x) - p(x)\|^2 = \int_a^b [f(x) - p(x)]^2 dx$$

E is then minimized with respect to every parameter in $p(x)$.

Example 5: Approximate $f(x) = e^x$ by the parabola

$$p(x) = ax^2 + bx + c,$$

on the interval $[0, 1]$.

Let

$$E = \int_0^1 (f(x) - p(x))^2 dx.$$

E is a minimum with respect to a, b and c if

$$\frac{\partial E}{\partial c} = 0, \quad \frac{\partial E}{\partial b} = 0, \quad \frac{\partial E}{\partial a} = 0.$$

This leads to the following set of equations in a, b and c:

$$\begin{array}{ccccccc} a\int_0^1 x^2 dx & + & b\int_0^1 x dx & + & c\int_0^1 1 dx & = & \int_0^1 e^x dx, \\ a\int_0^1 x^3 dx & + & b\int_0^1 x^2 dx & + & c\int_0^1 x dx & = & \int_0^1 xe^x dx, \\ a\int_0^1 x^4 dx & + & b\int_0^1 x^3 dx & + & c\int_0^1 x^2 dx & = & \int_0^1 x^2 e^x dx, \end{array}$$

or

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} e-1 \\ 1 \\ e-2 \end{bmatrix}. \tag{3.9}$$

with solution $a = 0.8392$, $b = 0.8511$ and $c = 1.0130$.

Figure 5 shows both $f(x) = e^x$ and $p(x) = ax^2 + bx + c$ on the same axes. Note how well $p(x)$ approximates $f(x)$, on the interval $[0, 1]$ and how the approximation quickly deteriorates outside the interval.

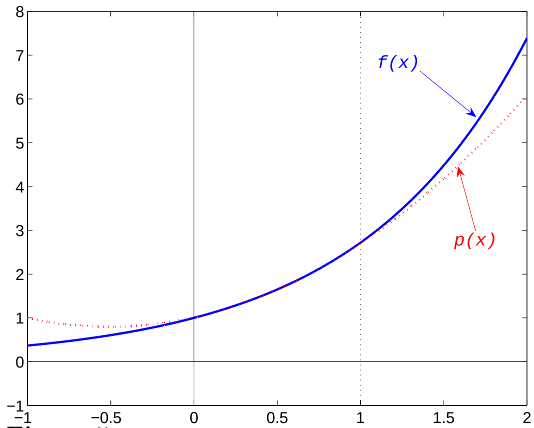

Figure 5.

3.2.4 Approximation with orthogonal functions

[In Example 5, equation (3.9), the matrix on the left was full and the full system of three equations in three unknowns had to be solved. Is there a way to simplify the system, for example, if the matrix is diagonal ?]

The answer is: If $p(x)$ is expressed as a combination of orthogonal functions, then the matrix is diagonal, and the system is easy to solve.]

Although orthogonality is defined for complex functions, we shall simplify the ideas in this section by only considering real functions.

Let $\{\phi_0(x), \phi_1(x), ..., \phi_N(x)\}$ be an orthogonal set of functions on the interval $[a, b]$:

$$\int_a^b \phi_j(x)\phi_k(x)dx = \begin{cases} 0 & \text{when} \quad j \neq k \\ \|\phi_k\|^2 & \text{when} \quad j = k \end{cases} \tag{3.10}$$

Let the approximating function be,

$$p(x) = \sum_{j=0}^{N} c_j \phi_j(x) \tag{3.11}$$

Let

$$E = \int_a^b [f(x) - p(x)]^2 dx$$

minimize E with respect to every c_k, $k = 0, 1, ..., N$:

$$\begin{aligned}
\frac{\partial E}{\partial c_k} &= \frac{\partial}{\partial c_k} \int_a^b \Big[f(x) - c_0\phi_0(x) - c_1\phi_1(x) - ... - c_N\phi_N(x)\Big]^2 dx \\
&= 2\int_a^b \Big[f(x) - c_0\phi_0(x) - c_1\phi_1(x) - ... - c_N\phi_N(x)\Big] \times \Big(-\phi_k(x)\Big)\, dx \\
&= -2\int_a^b f(x)\phi_k(x)dx + 2c_0\int_a^b \phi_0(x)\phi_k(x)dx + ... + 2c_k\int_a^b \phi_k(x)\phi_k(x)dx + ... \\
&= -2\int_a^b f(x)\phi_k(x)dx + 0 + 0 + ... + 2c_k\|\phi_k\|^2 + ... \\
&= 0
\end{aligned}$$

therefore

$$c_k = \frac{1}{\|\phi_k\|^2}\int_a^b f(x)\phi_k(x)dx. \tag{3.12}$$

This expression was derived by minimizing by means of derivatives. A different approach follows from the (geometrical) observation that $\|f(x) - p(x)\|$ is a minimum if its inner product with every $\phi_j(x)$ is zero. This approach gives the same result (3.12).

Example 6: Consider the following orthogonal functions on $[-1, 1]$:

$$\begin{aligned}
\phi_0(x) &= 1 & \|\phi_0\|^2 &= 2 \\
\phi_1(x) &= x & \|\phi_1\|^2 &= 2/3 \\
\phi_2(x) &= x^2 - \tfrac{1}{3} & \|\phi_2\|^2 &= 8/45
\end{aligned}$$

Find the least-squares approximation of the form

$$p(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x)$$

for $f(x) = e^x$ on the interval $[-1, 1]$.

Solution:

$$
\begin{aligned}
c_0 &= \frac{1}{\|\phi_0\|^2}\int_{-1}^{1} f(x)\phi_0(x)dx \\
&= \frac{1}{2}\int_{-1}^{1} e^x dx = \frac{e-e^{-1}}{2} = 1.1752 \\
c_1 &= \frac{1}{\|\phi_1\|^2}\int_{-1}^{1} f(x)\phi_1(x)dx \\
&= \frac{3}{2}\int_{-1}^{1} xe^x dx = 3e^{-1} = 1.1036 \\
c_2 &= \frac{1}{\|\phi_2\|^2}\int_{-1}^{1} f(x)\phi_2(x)dx \\
&= \frac{45}{8}\int_{-1}^{1} (x^2-\frac{1}{3})e^x dx = \frac{15e-105e^{-1}}{4} = 0.5367
\end{aligned}
$$

The least-squares approximation is

$$
\begin{aligned}
p(x) &= 1.1752 + 1.1036x + 0.5367(x^2 - \tfrac{1}{3}) \\
&= 0.5367x^2 + 1.1036x + 0.9963
\end{aligned}
$$

Figure 6 shows graphs of $f(x) = e^x$ as well as $p(x)$ on the same axes. The parabola approximates the function well on the interval, but the approximation deteriorates outside the interval.

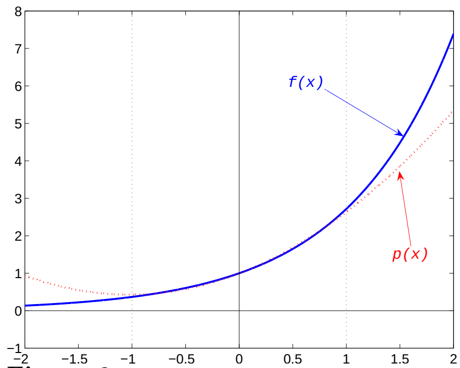

Figure 6.

Example 6: Check that the following set of functions are orthogonal on the interval $[-\pi, \pi]$:

$$\phi_0(x) = 1, \quad \phi_1(x) = \cos(x), \quad \phi_2(x) = \cos(2x)$$

Find the least-squares approximation of $f(x) = x^2$ on $[-\pi, \pi]$.
Answer: $c_0 = \pi^2/3, \quad c_1 = -4, \quad c_2 = 1.$
Figure 7 shows graphs of $p(x)$ and $f(x)$ on the same axes.

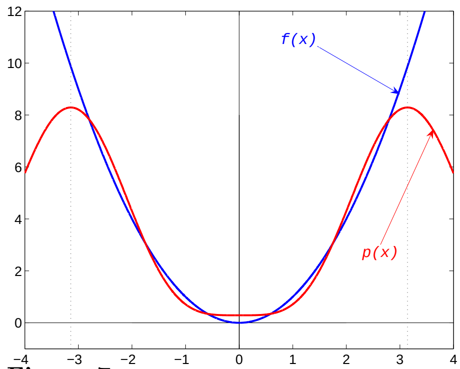

Figure 7.

Example 7: Take the same set of orthogonal functions (of Example 6) and find the least squares solution of $f(x) = \sin^2 x$ on $[-\pi, \pi]$. Draw graphs of the function and its approximation. Can you explain the result!

3.3 The Fourier series with an orthogonal basis

3.3.1 Orthogonality of the Fourier functions

[*In this section we show that the sine and cosine functions used in the Fourier series also form an orthogonal set.*]

Some identities:

You are reminded of the following identities, that will be needed subsequently.

$$\sin k\pi = 0 \qquad \text{for } k \in \mathbb{Z}$$

$$\cos k\pi = (-1)^k \qquad \text{for } k \in \mathbb{Z}$$

$$\cos A \cos B = \tfrac{1}{2}[\cos(A-B) + \cos(A+B)]$$

$$\sin A \sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\sin A \cos B = \tfrac{1}{2}[\sin(A-B) + \sin(A+B)]$$

The Fourier basis functions:

Consider the following set of functions, $\{\phi_k(x)\}$, $k = 0, 1, ...2N$ with

$$\begin{aligned} \phi_0(x) &= 1 \\ \phi_{2k}(x) &= \cos(k\pi x/L) \\ \phi_{2k-1}(x) &= \sin(k\pi x/L) \end{aligned}$$

We shall show that this set is orthogonal on the interval $[-L, L]$.

First check that the even functions (i.e. the cos-functions and 1) are orthogonal.

$$\begin{aligned}
&\int_{-L}^{L} \phi_{2j}(x)\phi_{2k}(x)dx \\
&= \int_{-L}^{L} \cos(j\pi x/L)\cos(k\pi x/L)dx \\
&= \begin{cases} \int_{-L}^{L} \frac{1}{2}(\cos(j\pi x/L - k\pi x/L) + \cos(j\pi x/L + k\pi x/L))dx & \text{if} \quad j \neq k \\ \int_{-L}^{L} \frac{1}{2}(1+\cos(2k\pi x/L))dx & \text{if} \quad j = k \neq 0 \\ \int_{-L}^{L} 1dx & \text{if} \quad j = k = 0 \end{cases} \\
&= \begin{cases} \frac{1}{2}\left[\frac{\sin((j-k)\pi x/L)}{(j-k)\pi/L} + \frac{\sin((j+k)\pi x/L)}{(j+k)\pi/L}\right]_{-L}^{L} & \text{if} \quad j \neq k \\ \frac{1}{2}\left[x + \frac{\sin(2\pi kx/L)}{2\pi k/L}\right]_{-L}^{L} & \text{if} \quad j = k \neq 1 \\ \Big[x\Big]_{-L}^{L} & \text{if} \quad j = k = 0 \end{cases} \\
&= \begin{cases} 0 & \text{if} \quad j \neq k \\ L & \text{if} \quad j = k \neq 0 \\ 2L & \text{if} \quad j = k = 0 \end{cases}
\end{aligned}$$

Next we check that the odd functions (i.e. the sines) are orthogonal:

$$\begin{aligned}
&\int_{-L}^{L} \phi_{2j-1}(x)\phi_{2k-1}(x)dx \\
&= \int_{-L}^{L} \sin(j\pi x/L)\sin(k\pi x/L)dx \\
&= \begin{cases} \int_{-L}^{L} \frac{1}{2}(\cos(j\pi x/L - k\pi x/L) - \cos(j\pi x/L + k\pi x/L))dx & \text{if} \quad j \neq k \\ \int_{-L}^{L} \frac{1}{2}(1-\cos(2k\pi x/L))dx & \text{if} \quad j = k \end{cases} \\
&= \begin{cases} \frac{1}{2}\left[\frac{\sin((j-k)\pi x/L)}{(j-k)\pi/L} - \frac{\sin((j+k)\pi x/L)}{(j+k)\pi/L}\right]_{-L}^{L} & \text{if} \quad j \neq k \\ \frac{1}{2}\left[x - \frac{\sin(2k\pi x/L)}{2k\pi/L}\right]_{-L}^{L} & \text{if} \quad j = k \end{cases} \\
&= \begin{cases} 0 & \text{if} \quad j \neq k \\ L & \text{if} \quad j = k \end{cases}
\end{aligned}$$

Two functions with opposite parity (i.e. one is odd, the other is even) are obviously orthogonal on $[-L, L]$, so that we do not need to check this by integrating out.

To summarize, this set of functions has the following property,

$$\int_{-L}^{L} \phi_j(x)\phi_k(x)dx = \begin{cases} 0 & j \neq k \\ L & j = k \neq 0 \\ 2L & j = k = 0 \end{cases}$$

Figure 8 shows the first seven Fourier functions.

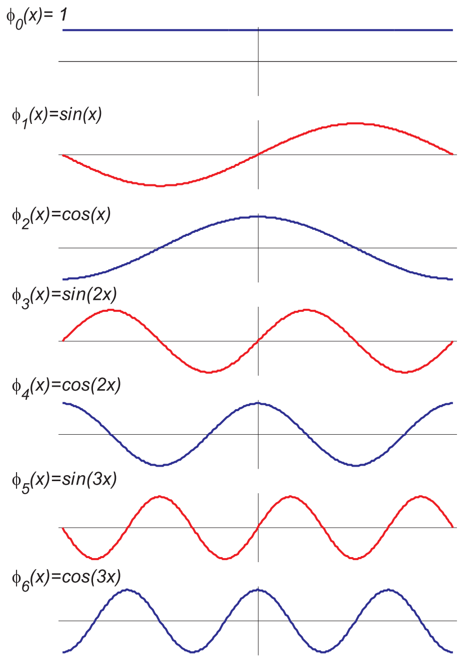

Figure 8. The first seven Fourier basis functions.

3.3.2 Fourier series again

[*Two concepts have been established now: (1) The Fourier functions form an orthogonal set, (2) Orthogonal functions can be used to approximate a given function over the interval of orthogonality. Let us now put these two ideas together.*]

For a given fixed N, we consider the space of all functions of the form

$$f_N(x) = \sum_{k=0}^{2N} c_k\phi_k(x) \tag{3.13}$$

$f_N(x)$ is sometimes called a *trigonometric polynomial of degree* N.

We may want to approximate a given function $f(x) : [-L, L] \to \mathbb{R}$, by $f_N(x)$.

If $f_N(x)$ is the least-squares approximation to $f(x)$ on $[-L, L]$, the theory derived in section 2.4 shows that the coefficients of the least-squares approximation are found from

$$c_k = \frac{1}{L}\int_{-L}^{L} f(x)\phi_k(x)dx \qquad \text{for} \quad k = 1, ..., 2N \tag{3.14}$$

and

$$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \tag{3.15}$$

Unfortunately the fact that $\|\phi_0\|^2 = 2L$ instead of L like all the other basis functions, force us to write it down separately.

A minor point in use of notation: The variable x in (3.14) is simply an integration variable, and in order to prevent confusion with the x used in (3.13), it may be replaced by any other symbol. For example, we may use ξ as integration variable, and write

$$c_k = \frac{1}{L}\int_{-L}^{L} f(\xi)\phi_k(\xi)d\xi \qquad \text{for} \quad k = 1, 2, ..., 2N \tag{3.16}$$

A more convenient way to express the series in (3.13), is to relabel the coefficients so that the index is the same as the frequency, and so that each type of function, sine and cosine, has its own coefficient.

Let

$$\begin{aligned} a_0 &= 2c_0 \\ \\ a_k &= c_{2k} && k = 1, 2, ..., N \\ b_k &= c_{2k-1} && k = 1, 2, , ..., N \end{aligned}$$

and then we express $f_N(x)$ as

$$\boxed{f_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(kx) + \sum_{k=1}^{N} b_k \sin(kx).} \tag{3.17}$$

Then

$$\boxed{\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(\xi)d\xi \\ a_k &= \frac{1}{L}\int_{-L}^{L} f(\xi)\cos(k\pi\xi/L)d\xi && k = 1, 2, ..., N \\ b_k &= \frac{1}{L}\int_{-L}^{L} f(\xi)\sin(k\pi\xi/L)d\xi && k = 1, 2, ..., N \end{aligned}} \tag{3.18}$$

and the Fourier series representation of $f(x)$ is expressed as

$$\begin{aligned} f_N(x) = \tfrac{1}{2}\int_{-L}^{L} f(\xi)d\xi &+ \sum_{k=1}^{N}\left[\int_{-L}^{L} f(\xi)\cos(k\pi\xi\mathrm{L})d\xi\right]\cos(kx) \\ &+ \sum_{k=1}^{N}\left[\int_{-L}^{L} f(\xi)\sin(k\pi\xi/L)d\xi\right]\sin(kx) \end{aligned} \tag{3.19}$$

Then $f_N(x)$ approximates $f(x)$ on the interval $[-L, L]$. However, note that $f_N(x)$ is actually a periodic function with period 2π. On the real axis, $x \in (-\infty, \infty)$, $f_N(x)$ approximates the *periodic continuation* of $f(x)$.

The coefficients $\{a_k, k = 0, 1, ...\}$ and $\{b_k, k = 1, 2, ...\}$ are called the *Fourier coefficients*.

When N is finite, we refer to (3.17) as a "Fourier approximation" or "truncated Fourier series" of $f(x)$. If $N \to \infty$, then (3.17) is called a "Fourier series".

3.3.3 The Complex Fourier Series

By using complex variables, a more symmetric and compact formula may be obtained for the Fourier series.

You are reminded of the following identities:

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta \tag{3.20}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{3.21}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{3.22}$$

Let $\{a_k\}$ and $\{b_k\}$ be the Fourier coefficients of the function $f(x)$ defined on $[-L, L]$. We introduce a new set of coefficients, viz. $\{c_k\}, k = -N, -N+1, ..., -1, 0, 1, ..., N$. These are given by

$$\left.\begin{array}{rcl} c_0 & = & a_0 \\ & & \\ c_k & = & \frac{1}{2}(a_k - ib_k) \\ & & \\ c_{-k} & = & \frac{1}{2}(a_k + ib_k) \end{array}\right\} \quad \text{vir} \quad k = 1, 2, ..., N \tag{3.23}$$

The original coefficients may be found from $\{c_k\}$ with the following formulae

$$\left.\begin{array}{rcl} a_k & = & c_k + c_{-k} \\ & & \\ b_k & = & i(c_k - c_{-k}) \end{array}\right\} \quad \text{vir} \quad k = 1, 2, ..., N \tag{3.24}$$

and $a_0 = c_0$.

The Fourier series may then be expressed as

$$\begin{aligned} f_N(x) &= a_0 + \sum_{k=1}^{N} \left[a_k \cos(k\pi x/L) + b_k \sin(k\pi x/L)\right] \\ &= c_0 + \sum_{k=1}^{N} \left[(c_k + c_{-k})\cos(k\pi x/L) + i(c_k - c_{-k})\sin(k\pi x/L)\right] \\ &= c_0 + \sum_{k=1}^{N} \left[c_k\left(\cos(k\pi x/L) + i\sin(k\pi x/L)\right) + c_{-k}\left(\cos(k\pi x/L) - i\sin(k\pi x/L)\right)\right] \\ &= c_0 e^0 + \sum_{k=1}^{N} \left[c_k e^{ik\pi x/L} + c_{-k} e^{-ik\pi x/L}\right] \end{aligned}$$

and therefore

$$\boxed{f_N(x) = \sum_{k=-N}^{N} c_k e^{ik\pi x/L}} \tag{3.25}$$

A new formula for the Fourier coefficients can be obtained as follows:

$$\begin{aligned}
c_{\pm k} &= \tfrac{1}{2}(a_k \mp ib_k) \\
&= \frac{1}{2}\left[\frac{1}{\pi}\int_{-L}^{L} f(\xi)\cos(k\pi\xi/L)d\xi \;\mp\; \frac{i}{L}\int_{-L}^{L} f(\xi)\sin(k\pi\xi/L)d\xi\right] \\
&= \frac{1}{2L}\int_{-L}^{L} f(\xi)(\cos(k\pi\xi/L) \;\mp\; i\sin(k\xi))d\xi \\
&= \frac{1}{2L}\int_{-L}^{L} f(\xi)e^{\mp ik\pi\xi/L}d\xi
\end{aligned} \tag{3.26}$$

and

$$c_0 = a_0 = \frac{1}{2L}\int_{-L}^{L} f(\xi)e^0 d\xi$$

and therefore the case $k = 0$ is already covered by (3.26).

The formula for the complex Fourier coefficients is therefore

$$\boxed{c_k = \frac{1}{2L}\int_{-L}^{L} f(\xi)e^{-ik\pi\xi/L}d\xi \quad \text{for } k = -N, ..., N} \tag{3.27}$$

In this formulation $f(x)$ may even be a complex function of a real variable x.

An alternative way to establish the complex Fourier series, is to consider the set of functions

$$\{\phi_k(x) = e^{ik\pi x/L}, \quad k \in \mathbb{Z}\}$$

and show that they are orthogonal on the interval $[-L, L]$. The result is as follows,

$$\int_{-L}^{L} \overline{\phi_k(x)}\phi_j(x)dx = \begin{cases} 0, & k \neq j, \\ 2L, & k = j. \end{cases} \tag{3.28}$$

and the formula for the Fourier approximation of $f(x)$ on $[-L, L]$ follows from (3.28) and (3.12).

Example 8: Find the complex Fourier series of $f(x) = e^x$, on $[-2, 2]$.

$$\begin{aligned}
c_k &= \frac{1}{2\times 2}\int_{-2}^{2} e^x e^{-ik\pi x/2}dx \\
&= \frac{1}{4}\left.\frac{e^{x(1-ik\pi/2)}}{(1-ik\pi/2)}\right|_{-2}^{2} \\
&= \frac{1}{2}\left(\frac{e^{2-ik\pi} - e^{-2+ik\pi}}{2-ik\pi}\right).
\end{aligned}$$

and the Fourier series is

$$f_\infty(x) = \frac{1}{2}\sum_{k=-\infty}^{\infty}\left(\frac{e^{2-ik\pi}-e^{-2+ik\pi}}{2-ik\pi}\right)e^{ik\pi x/2}.$$

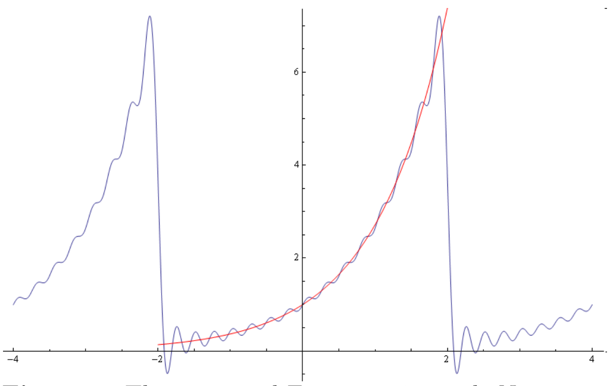

Figure 9. The truncated Fourier series with $N = 15$.

3.4 Half and Quarter series

For a function defined on $x \in [-L, L]$ (the 'full interval') the cos+sin Fourier series is straight forward. One must calculate the a_k's and the b_k's in the normal way.

However, in many applications a function $f(x)$ is supplied on only some part of the interval, and it is required that some of the Fourier coefficients must be zero. This means that the function must be *completed* by using copies or reflections of $f(x)$ to fill up the rest of the interval. The most important series of this type are the so-called 'half series' and 'quarter series'.

3.4.1 Fourier half series

The 'half series' is simple: The function $f(x)$ is given on $x \in [0, L]$, and the Fourier series must have either only sine terms, or only cosine terms.

Case (A): Only sine terms.

The function is given on $x \in [0, L]$, and

$$f(x) = \sum_{k=1}^{\infty} b_k \sin(k\pi x/L).$$

Since only sine terms occur in the series, it is the series of an *odd* function. Therefore $f(x)$, must be completed on the interval $[-L, L]$ as an odd function. Therefore

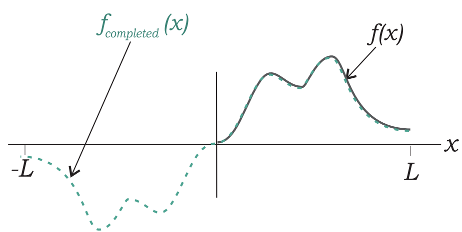

Figure 10. An odd completion of $f(x)$.

$$f_{completed}(x) = \begin{cases} f(x), & x \in [0, L], \\ -f(-x), & x \in [-L, 0]. \end{cases}$$

Figure 10 shows $f(x)$ as well as $f_{completed}(x)$.

Just note that it is not necessary to integrate over the whole interval. Because $f_{completed}$ is odd, we have

$$b_k = \frac{2}{L} \int_0^L f(x) \sin(k\pi x/L) dx.$$

Case (B): Only cosine terms.

The function is given on $x \in [0, L]$, and

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x/L).$$

Since only cosine terms occur in the series, it is the series of an *even* function. Therefore $f(x)$, must be completed on the interval $[-L, L]$ as an even function. Then

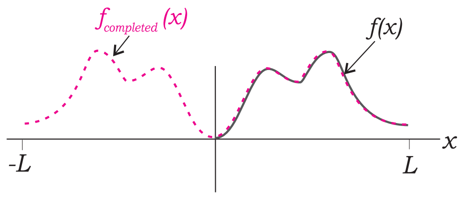

Figure 11. An even completion of $f(x)$.

$$f_{completed}(x) = \begin{cases} f(x), & x \in [0, L], \\ f(-x), & x \in [-L, 0]. \end{cases}$$

Figure 11 shows $f(x)$ and $f_{completed}(x)$.

Once again only integration over $[0, L]$ is needed,

$$a_k = \frac{2}{L} \int_0^L f(x) \cos(k\pi x/L) dx$$

3.4.2 Fourier quarter series

With the 'quarter series' only one quarter of the function is supplied and it is required that its Fourier series has only sine or cosine terms and in addition there is the requirement that all the even coefficients should be zero (or alternatively, all the odd coefficients should be zero.)

We shall illustrate this with one example.

Example:

The function $f(x)$ is given on $x \in [0, L]$, and it is required that its Fourier series must be of the following form:

$$f(x) = \sum_{k=1, k \text{ odd}}^{\infty} a_k \cos\left(\frac{k\pi x}{2L}\right)$$

Since only cosine terms occur in the series, it is the series of an even function. Since the arguments of the cosine functions are of the form $k\pi x/(2L)$, the function is periodic on $[-2L, 2L]$, but the function is given only over $[0, L]$. This means that only *a quarter* of the full function is supplied. Figure 12 shows the setup.

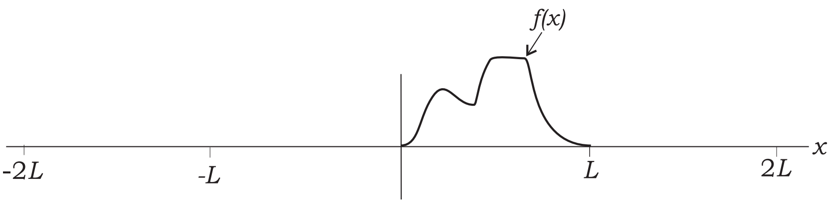

Figure 12. Only one quarter of the function is supplied.

It is clear that $f(x)$, must be completed on the interval $[-2L, 2L]$ as an *even* function. Let the part of the completed function between $[L, 2L]$ be $g(x)$, so that $f_{completed}$ is given by

$$f_{completed}(x) = \begin{cases} g(-x), & x \in [-2L, -L), \\ f(-x), & x \in [-L, 0), \\ f(x), & x \in [0, L), \\ g(x), & x \in [L, 2L). \end{cases}$$

We need to find the relationship between $f(x)$ and $g(x)$.

The Fourier coefficients are

2, because an even function is integrated only over half the interval.

$$a_k = \frac{2}{(2L)} \left[\int_0^L f(x) \cos\left(\frac{k\pi x}{2L}\right) dx + \int_L^{2L} g(x) \cos\left(\frac{k\pi x}{2L}\right) dx \right].$$

(2L), because the full period is now $4L$.

Consider the second integral

$$X = \int_L^{2L} g(x) \cos\left(\frac{k\pi x}{2L}\right) dx.$$

In order to convert the limits to $[0, L]$, we substitute $x = 2L - y$, into X then

$$
\begin{aligned}
X &= \int_L^0 g(2L-y)\cos\left(\frac{k\pi}{2L}(2L-y)\right)(-dy) \\
&= \int_0^L g(2L-y)\cos\left(\frac{k\pi}{2L}(2L-y)\right)dy \\
&= \int_0^L g(2L-y)\cos\left(\frac{-k\pi y}{2L}+k\pi\right)dy \\
&= \int_0^L g(2L-y)\left[\cos\left(\frac{-k\pi y}{2L}\right)\cos(k\pi)-\sin\left(\frac{-k\pi y}{2L}\right)\sin(k\pi)\right]dy \\
&= (-1)^k\int_0^L g(2L-y)\cos\left(\frac{k\pi y}{2L}\right)dy
\end{aligned}
$$

using x again as integration variable....

$$= (-1)^k\int_0^L g(2L-x)\cos\left(\frac{k\pi x}{2L}\right)dx.$$

Then

$$a_k = \frac{1}{L}\int_0^L \Big(f(x)+(-1)^k g(2L-x)\Big)\cos\left(\frac{k\pi x}{2L}\right)dx.$$

We now impose the requirement that all a_k be zero, for even k.

$$0 = \frac{1}{L}\int_0^L \Big(f(x)+g(2L-x)\Big)\cos\left(\frac{k\pi x}{2L}\right)dx,$$

This can only be valid for all k, if

$$f(x)+g(2L-x)=0, \qquad \text{for all } x,$$

that is,

$$g(x)=-f(2L-x), \qquad \text{for all } x.$$

Then $g(x)$ is just $f(x)$ that is flipped in the x-direction as well as in the y-direction and shifted to the interval $[L, 2L]$.

Figure 13 shows shows $f(x)$ and $f_{completed}(x)$.

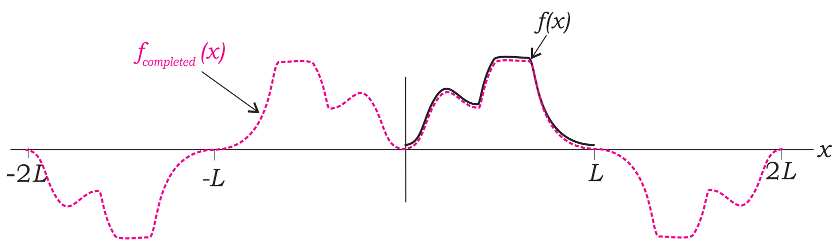

Figure 13. The completion of $f(x)$ on $[-2L, 2L]$.

Once again only integration over $[0, L]$ is needed, since for odd k, we have

$$a_k = \frac{1}{L}\int_0^L \Big(f(x) - g(2L-x)\Big)\cos\left(\frac{k\pi x}{2L}\right)dx],$$

and substituting $g(2L - x) = -f(x)$, gives

$$a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi x}{2L}\right)dx.$$

There are four types of quarter series. The formulae for each may be derived as in the above example. The formulae are listed in the table below and typical graphs are shown in Figure 14.

(A)	$f_{cmpl}(x) = \begin{cases} -f(2L+x), & x \in [-2L, -L) \\ +f(-x), & x \in [-L, 0), \\ +f(x), & x \in [0, L), \\ -f(2L-x), & x \in [L, 2L). \end{cases}$	$f(x) = a_1 \cos\left(\frac{\pi x}{2L}\right) + a_3 \cos\left(\frac{3\pi x}{2L}\right) + a_5 \cos\left(\frac{5\pi x}{2L}\right) + ...$
(B)	$f_{cmpl}(x) = \begin{cases} +f(2L+x), & x \in [-2L, -L) \\ +f(-x), & x \in [-L, 0), \\ +f(x), & x \in [0, L), \\ +f(2L-x), & x \in [L, 2L). \end{cases}$	$f(x) = a_2 \cos\left(\frac{2\pi x}{2L}\right) + a_4 \cos\left(\frac{4\pi x}{2L}\right) + a_6 \cos\left(\frac{6\pi x}{2L}\right) + ...$
(C)	$f_{cmpl}(x) = \begin{cases} +f(2L+x), & x \in [-2L, -L) \\ -f(-x), & x \in [-L, 0), \\ +f(x), & x \in [0, L), \\ -f(2L-x), & x \in [L, 2L). \end{cases}$	$f(x) = b_2 \sin\left(\frac{2\pi x}{2L}\right) + b_4 \sin\left(\frac{4\pi x}{2L}\right) + b_6 \sin\left(\frac{6\pi x}{2L}\right) + ...$
(D)	$f_{cmpl}(x) = \begin{cases} -f(2L+x), & x \in [-2L, -L) \\ -f(-x), & x \in [-L, 0), \\ +f(x), & x \in [0, L), \\ +f(2L-x), & x \in [L, 2L). \end{cases}$	$f(x) = b_1 \sin\left(\frac{\pi x}{2L}\right) + b_3 \sin\left(\frac{3\pi x}{2L}\right) +_5 \sin\left(\frac{5\pi x}{2L}\right) + ...$

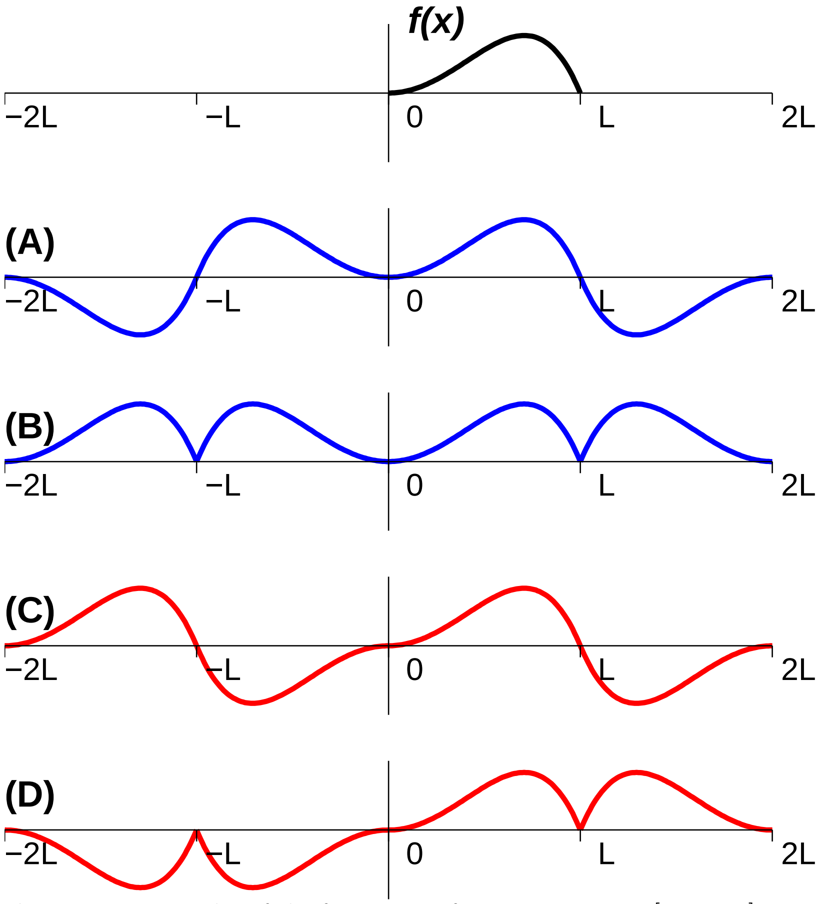

Figure 14. Examples of the four types of quarter series on $[-2L, 2L]$ using the same $f(x)$.