

Die toets gaan oor die volgende onderwerpe:

- EIEWAARDE-ONTBINDING (weer, veral die simmetriese geval)
- STELSELS VAN VERSKILVERGELYKINGS
- STELSELS VAN DIFFERENSIAALVERGELYKINGS
- EIEWAARDES VAN SIMMETRIESE Matrikse ($A = Q\Lambda Q^T$)
- KWADRATIESE KROMMES
- DIE SVD
- ROTASIES IN 3D

EIEWAARDES EN EIEVEKTORE

- Eiewaardes en eievektore van gegewe 2×2 en 3×3 matrikse kan vind, en veral weet hoe om dit makliker te vind as een eievektor of een eiewaarde gegee is. Ook die gebruik van $\det(A)$ en $\text{tr}(A)$, kan die vind van eiewaardes vergemaklik.
- Diagonaalvorm van 'n matriks kan vind, d.w.s. $A = S\Lambda S^{-1}$
- As A simmetries is, die spesiale diagonaalvorm met ortogonale matrikse kan vind, $A = Q\Lambda Q^T$, ook as A gelyke eiewaardes het.
- Die verband tussen die som en produk van die eiewaardes met die determinant en spoor van die matriks kan aantoon vir die 2×2 geval.
- Spesiale eienskappe van die eiewaardes en eievektore van *simmetriese* matrikse ken en kan aantoon, bv. eiewaardes is reëel en $(\lambda_k - \lambda_j)\mathbf{x}_k^T \mathbf{x}_j = 0$ en dit kan interpreteer.
- Die verband wat die eiewaardes en eievektore van A , A^n , A^{-1} en $A - \alpha I$ met mekaar het ken en kan aantoon.

The test will be on the following topics

- EIGENVALUE DECOMPOSITION (again, especially the symmetric case)
- SYSTEMS OF DIFFERENCE EQUATIONS
- SYSTEMS OF DIFFERENTIAL EQUATIONS
- EIGENVALUES OF SYMMETRIC MATRICES ($A = Q\Lambda Q^T$)
- QUADRATIC CURVES
- THE SVD
- ROTATIONS in 3D

EIGENVALUES AND EIGENVECTORS

- Be able to find eigenvalues and eigenvectors of a given 2×2 and 3×3 matrix, especially how to find it easier if one eigenvector or one eigenvalue is given. The use of $\det(A)$ and $\text{tr}(A)$ can also simplify the finding of eigenvalues.
- Be able to find the diagonal form of a matrix, i.e. $A = S\Lambda S^{-1}$
- If A is symmetric, be able to find the special diagonal form with orthogonal matrices, $A = Q\Lambda Q^T$, also if A has equal eigenvalues.
- Show the relationship between the sum and product of the eigenvalues, and the determinant and trace of the matrix, for the 2×2 case.
- Know and derive special properties of the eigenvectors and eigenvalues of symmetric matrices, e.g. eigenvalues are real and $(\lambda_k - \lambda_j)\mathbf{x}_k^T \mathbf{x}_j = 0$ and be able to interpret it.
- Know and derive the relationship that the eigenvalues and eigenvectors of A , A^n , A^{-1} and $A - \alpha I$ have with each other.

STELSELS VAN VERSKILVERGELEYKINGS

- Herlei die oplossing van 'n stelsel van verskilvergelykings van die vorm $\mathbf{u}_{n+1} = A\mathbf{u}_n$. (Dit is $\mathbf{u}_k = S\Lambda^k S^{-1}\mathbf{u}_0$.)
- Stelsels van eerste orde verskilvergelykings kan oplos (slegs 2×2 -stelsels).
- Tweede orde verskilvergelykings kan skryf as stelsels van eerste orde verskilvergelykings en kan oplos.
- Vir praktiese probleme wat lei tot 'n stelsel van verskilvergelykings, die stelsel kan neerskryf en uitdruk in matriksvorm. (Diskrete groei-modelle, getalle-reekse, ens.)
- Limiete kan vind as $k \rightarrow \infty$, waar van toepassing.

STELSELS VAN DIFFERENSIAALVERGELYKINGS

- Herlei die oplossing van 'n stelsel van differensiaalvergelykings van die vorm $\dot{\mathbf{u}} = A\mathbf{u}$. (Dit is $\mathbf{u}(t) = S e^{\Lambda t} S^{-1}\mathbf{u}(0)$.)
- Stelsels van eerste orde differensiaalvergelykings kan oplos (slegs 2×2 -stelsels).
- Tweede orde differensiaalvergelykings kan skryf as stelsels van eerste orde differensiaalvergelykings en kan oplos.
- In praktiese probleme wat 'n toepassing van differensiaalvergelykings is, die stelsel kan neerskryf en ook uitdruk in matriksvorm. (Tenks met oplossings in., Meganiese stelsels, ens. Genoeg wenke sal gegee word.)
- Limiete kan vind as $t \rightarrow \infty$, maksima of minima van oplossings kan vind, of die tyd wanneer 'n maksimum of minimum voorkom kan vind.

SYSTEMS OF DIFFERENCE EQUATIONS

- Derive the solution of a system of difference equations of the form $\mathbf{u}_{n+1} = A\mathbf{u}_n$. (It is $\mathbf{u}_k = S\Lambda^k S^{-1}\mathbf{u}_0$.)
- Solve systems of first order difference equations (2×2 -systems only).
- Be able to write second order difference equations as systems of first order difference equations and solve it.
- For practical problems that are expressible as a system of difference equation, you must be able to write down the system and express it in matrix form (Discrete growth models, number sequences, etc.)
- Be able to find limits as $k \rightarrow \infty$ where applicable.

SYSTEMS OF DIFFERENTIAL EQUATIONS

- Derive the solution of a system of differential equations of the form $\dot{\mathbf{u}} = A\mathbf{u}$. (It is $\mathbf{u}(t) = S e^{\Lambda t} S^{-1}\mathbf{u}(0)$.)
- Solve systems of first order differential equations (2×2 -systems only).
- Be able to write second order differential equations as systems of first order differential equations and solve it.
- In practical problems that are applications of differential equations, you must be able to write down the system and also express it in matrix form. (Tanks with solutions, mechanical systems, etc. Enough hints will be given.)
- Be able to find limits as $t \rightarrow \infty$, or find maxima or minima or find the time when a maximum or minimum occurs.

KWADRATIESE KROMMES

- Verduidelik hoe eiewaarde-ontbinding gebruik word om die hoofasse en rotasiehoek van 'n algemene geroteerde ellips of hiperbool te vind (m.a.w. die teorie).
- Moet die rotasiehoek en die lengtes van die semihoofasse van die figuur voorgesel deur 'n kwadratiese vergelyking, kan vind en 'n rowwe skets van die figuur kan maak met al die inligting daarop aangedui. Slegs ellipse of hiperbole kan gevra word.
- *Laat uit: geskuifde kwadratiese krommes.*

DIE SVD

- Moet $2 \times n$ en $n \times 2$ matrikse kan SVD-ontbind, met $n = 2, 3, 4$ of 5 . Beide die volle SVD of die gereduseerde SVD kan gevra word.
- Die interpretasie ken van wat elk van die faktore U , Σ en V^T doen as 'n matriks met 'n vektor vermenigvuldig word (roteer-skaal-roteer).
- Moet ortogonale basisse van die vier fundermentele ruimtes (kolom-, ry-, nul-, en linksnul-ruimte) van 'n matriks kan neerskryf as die SVD van 'n matriks beskikbaar is. Moet ook kan aantoon waar hierdie verband vandaan kom.
- Moet die verband tussen die SVD en rang-1 ontbinding van 'n matriks kan beskryf.
- Jy moet die kleinste-kwadrate oplossing van 'n reghoekige stelsel kan vind met behulp van die SVD.

QUADRATIC CURVES

- Explain how eigenvalue decomposition is used to obtain the semi-axes and rotation angle of a rotated ellipse or hyperbola (i.e. the theory).
- Find the rotation angle and the lengths of the semi-axes of the figure represented by a quadratic equation, and draw a rough sketch of the figure, containing this information. Only ellipses or hyperbolas will be asked.
- Omit: shifted quadratic curves.

THE SVD

- SVD-decompose $2 \times n$ and $n \times 2$ matrices, with $n = 2, 3, 4$ or 5 . Both the full SVD or the reduced SVD can be asked.
- Know the interpretation of what each of the factors U , Σ and V^T does, when a matrix is multiplied with a vector (rotate-scale-rotate).
- Write down the orthogonal basis of each of the four fundamental spaces (column, row, null, and left-nullspace) of a matrix, if the SVD of a matrix is available. Also be able to show where this relationship comes from.
- Must be able to describe the relationship between the SVD and the rank-1 decomposition of a matrix.
- You must be able to find the least squares solution of a rectangular system by using the SVD.

DIE 3×3 -ROTAASIE-MATRIKS

- Moet die verband tussen die kruisproduk-matriks A en die rotasie-as \mathbf{a} ken en kan neerskryf.
- Moet die eienskappe van A en A^2 kan aantoon (simmetrie, spoor, en $\mathbf{a}\mathbf{a}^T = I + A^2$).
- Moet die volgende verband tussen die rotasiematriks Q en die rotasie-as \mathbf{a} en hoek θ ken en kan herlei uit eerste beginsels: $Q = I + A \sin \theta + A^2(1 - \cos \theta)$.
- Moet die vorm $Q = I + A \sin \theta + A^2(1 - \cos \theta)$ kan gebruik om die eienskappe van Q aan te toon.
- Moet die eienskappe van die eiewaardes en eievektore van Q en A kan aantoon en kan interpreteer.
- Moet die formules $\cos \theta = \frac{1}{2}(\text{Tr}(Q) - 1)$ en $A = (Q - Q^T)/(2 \sin \theta)$ kan herlei, en kan gebruik om die rotasie-as en rotasie-hoek van 'n rotasie-matriks te kan vind. Die rotasie-matriks Q moet ook gevind kan word as θ en \mathbf{a} gegee is.
- Eenvoudige toepassings kan gevra word.

Spesiale insig-vrae

Vrae of gedeeltes van vrae wat nie presies in een van die kategorieë hierbo val nie, maar wat hoofsaaklik die kennis oor lineêre algebra toets wat jy in hierdie kursus geleer het, *saam met insig*, kan gevra word.

THE 3×3 ROTATION MATRIX

- Know and be able to write down the relationship between the cross-product matrix A and the rotation axis \mathbf{a} .
- Prove the properties of A and A^2 (symmetry, trace, and $\mathbf{a}\mathbf{a}^T = I + A^2$).
- Know and derive the following relationship between the rotation matrix Q and the rotation axis \mathbf{a} and the angle θ , from first principles: $Q = I + A \sin \theta + A^2(1 - \cos \theta)$.
- Must be able to use the form $Q = I + A \sin \theta + A^2(1 - \cos \theta)$ to prove properties of Q .
- Be able to derive and interpret the properties of the eigenvalues and eigenvectors of Q and A .
- Derive the formulae $\cos \theta = \frac{1}{2}(\text{Tr}(Q) - 1)$ and $A = (Q - Q^T)/(2 \sin \theta)$, and use them to find the rotation axis and rotation angle of a rotation matrix. Also, be able to obtain the rotation matrix if θ and \mathbf{a} are given.
- Simple applications may be asked.

Special insight questions

Questions or parts of questions that do not clearly fall into any of the above categories, but that mainly tests the knowledge of linear algebra that you acquired during this module, with insight, may be asked.