AM214-2023: LECTURE 36

LECTURE 36 — ROTATIONS IN 3D

1

$$\underline{a} \times \underline{b} = A\underline{b} \qquad A = \text{cpmat}(\underline{a})$$

$$\underline{a} \times \underline{b} = ab\sin\theta\, \hat{\underline{c}} \qquad = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$\|\underline{a}\| = 1$

- $A^T = -A$
- A^2 is symmetric
- $A\underline{a} = \underline{0}$ — $\underline{a}$ lies in nullspace; $\underline{a}$ is evec, with $\lambda = 0$
- $A^2 + I = \underline{a}\underline{a}^T$
- $A^3 = AA^2 = A(\underline{a}\underline{a}^T - I) = (\overset{0}{A\underline{a}})\underline{a}^T - A = -A$

$$A^4 = -A^2, \quad A^5 = A, \quad A^6 = A^2, \quad A^7 = -A \; \ldots$$

2

- $\text{tr}(A^2) = \text{tr}(\underline{a}\underline{a}^T - I)$

$$= \text{tr}\left(\begin{bmatrix} a_1a_1 & a_1a_2 & a_1a_3 \\ & a_2a_2 & \\ & & a_3a_3 \end{bmatrix}\right) - 3$$

$$= 1 - 3 = -2$$

- $\underline{b}^T A \underline{b} = 0$

ROTATION MATRIX Q

$$Q\underline{a} = 1\underline{a}$$

FORWARD — Given $\underline{a}$, θ, find Q

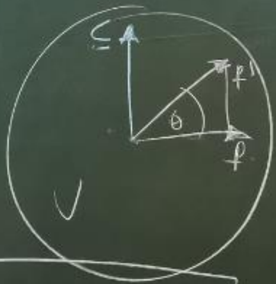
INVERSE — Given Q, find $\underline{a}$, θ.

$$\boxed{Q = S\Lambda S^{-1}, \quad \Lambda = \begin{bmatrix} 1 & & \\ & e^{i\theta} & \\ & & e^{-i\theta} \end{bmatrix}, \quad S = \begin{bmatrix} \underline{a} & \text{comp.} & \text{comp} \end{bmatrix}}$$

3

$\underline{a} \times \underline{b} = \underline{c}$

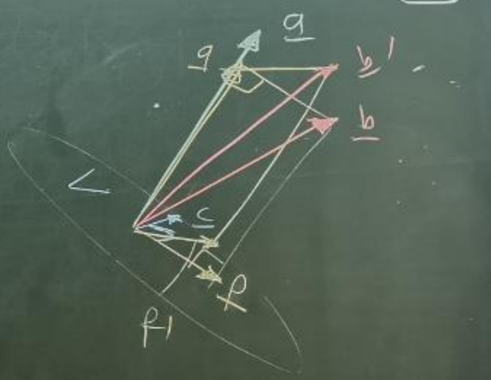
$\underline{p}$ and $\underline{c}$ lie in V



$$\boxed{p' = c_\theta p + s_\theta \underline{c}}$$

$\underline{b}' = \ldots\ldots$ in terms of $\underline{b}$.

$= (\ldots\ldots)\,\underline{b}$



$\underline{p}$ and $\underline{c}$ have same lengths.

$q = \underline{a}\,\underline{a}^T \underline{b}$

$\cancel{\underline{a}^T\underline{a}}$

$\underline{p} = \underline{b} - q = \underline{b} - \underline{a}\,\underline{a}^T \underline{b}$

$= (I - \underline{a}\,\underline{a}^T)\,\underline{b}$

4

$$\boxed{Q = I + s_\theta A + (1 - c_\theta) A^2}$$

Show $Q^T Q = I$

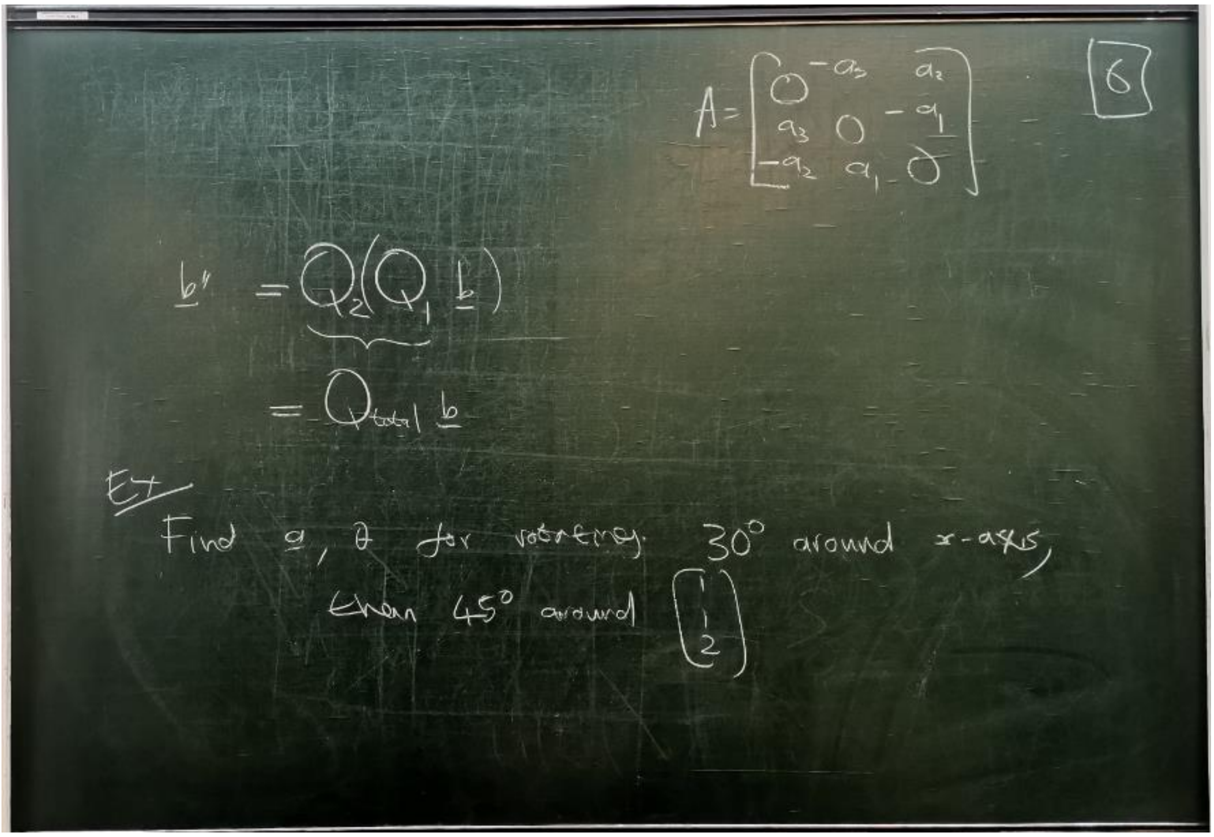
90° around x

90° around y

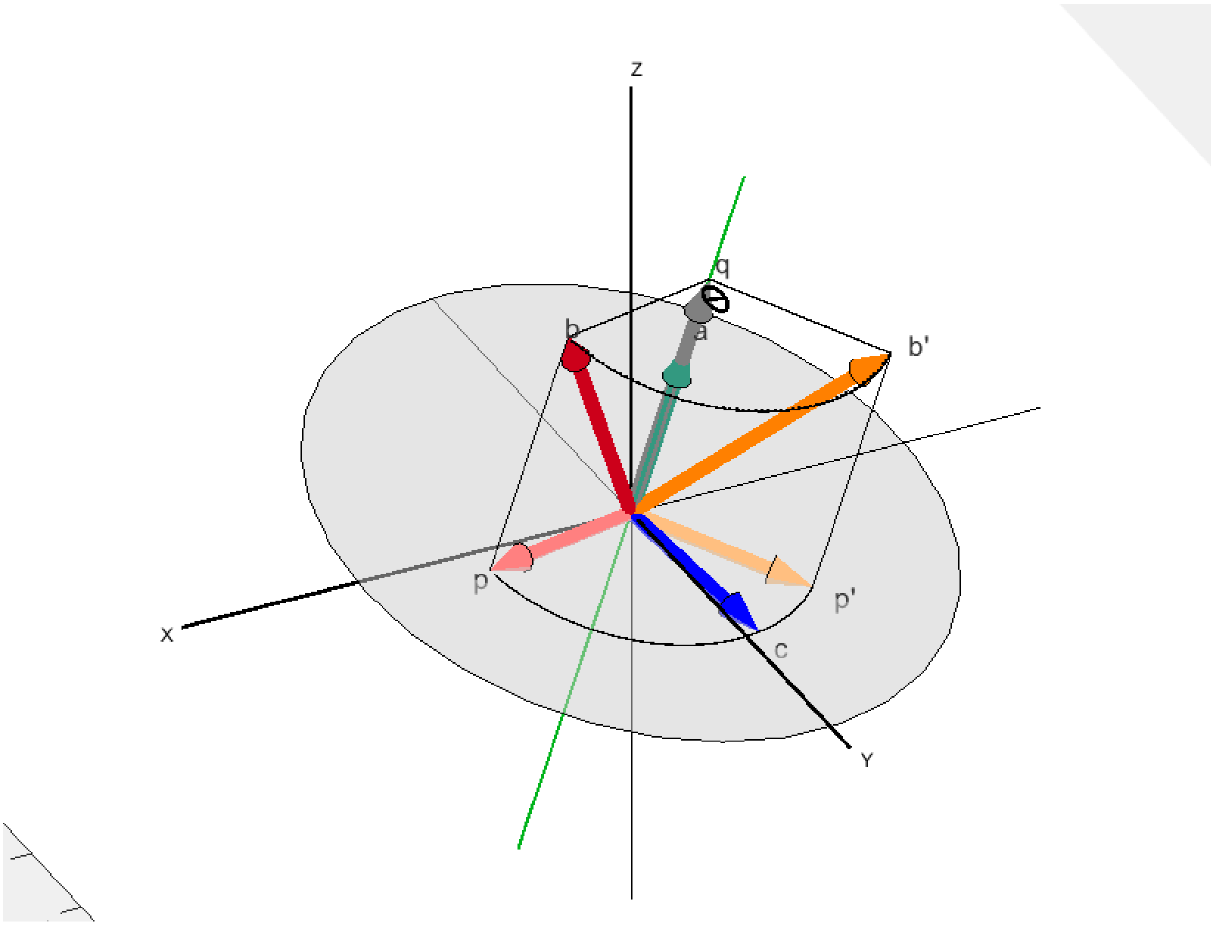
Rotations in 3D are not commutative.

$\underline{a}, \theta \longrightarrow Q = I + s_\theta A$

$Q \longrightarrow \underline{a}, \theta$



$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$
$\boxed{6}$
$\underline{b}'' = \underbrace{Q_2(Q_1}\, \underline{b})$
$= Q_{total}\, \underline{b}$
Ex
Find $\underline{a}$, θ for rotating $30°$ around x-axis,
then $45°$ around $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$



z
q
θ
b
a
b'
p
p'
c
x
Y