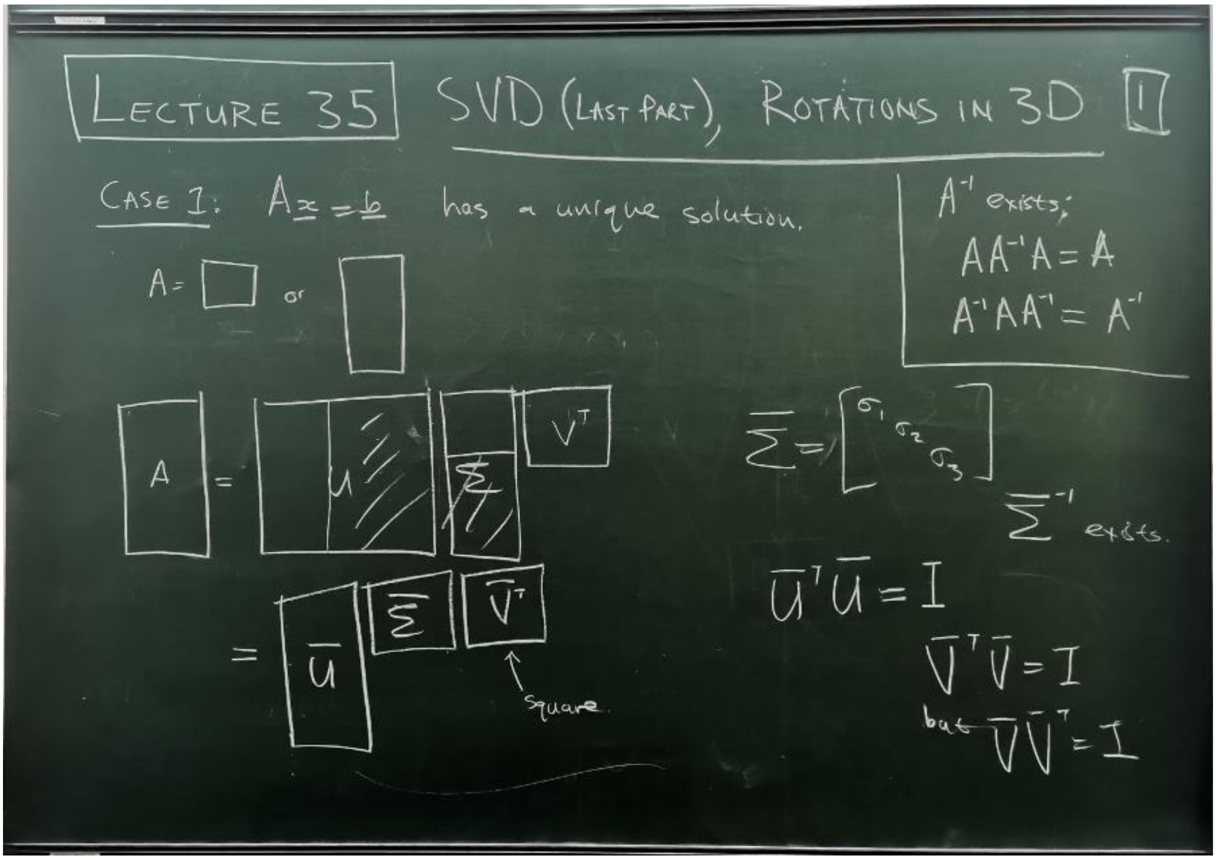
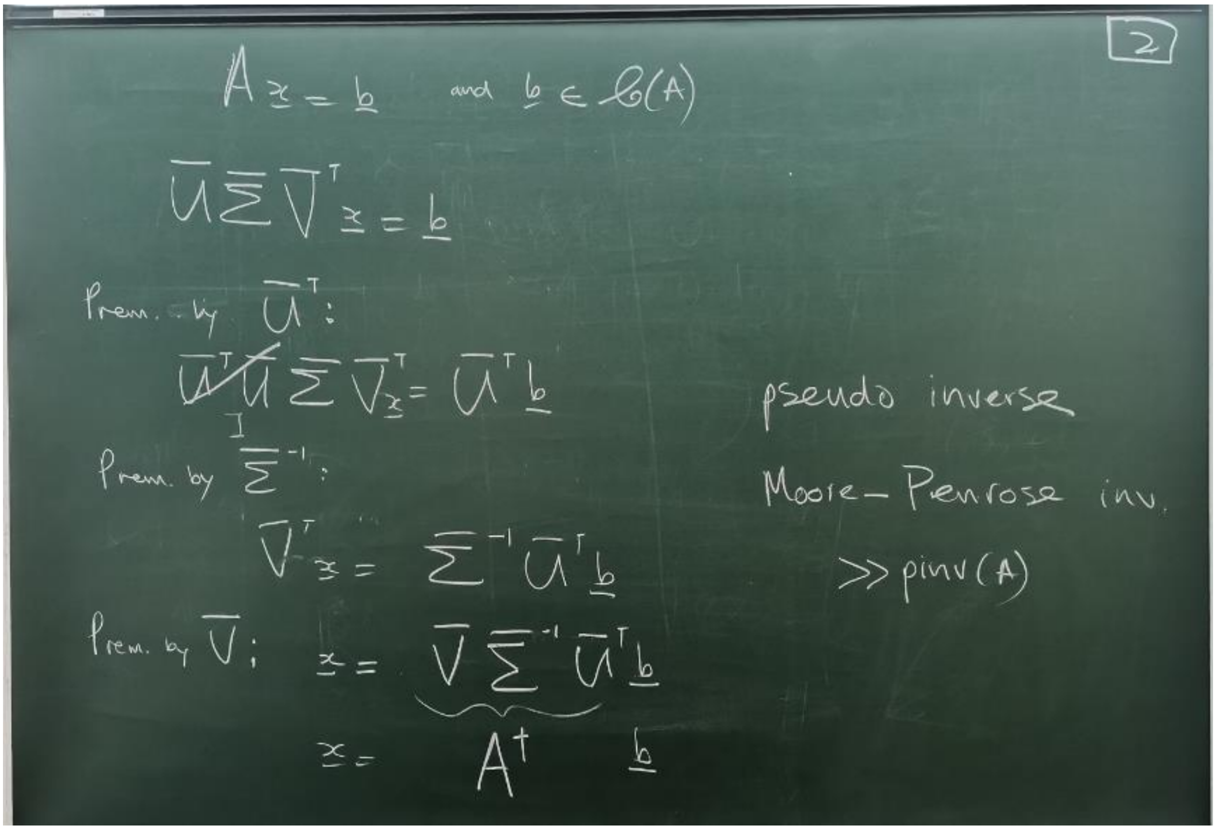
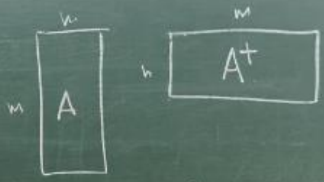
AM214-2023: LECTURE 35





Def: $A^\dagger = \bar{V}\bar{\Sigma}^{-1}\bar{U}^T$



Properties:

$$AA^\dagger A = A \qquad A^\dagger A A^\dagger = A^\dagger$$

$$(\bar{U}\bar{\Sigma}\bar{V}^T)(\bar{V}\bar{\Sigma}^{-1}\bar{U}^T)(\bar{U}\bar{\Sigma}\bar{V}^T)$$

$$= \bar{U}\bar{\Sigma}\bar{\Sigma}^{-1}\bar{\Sigma}\bar{V}^T = \bar{U}\bar{\Sigma}\bar{V}^T = A$$

Case 2: No solution, try LS-solution

$A\underline{x} = \underline{b} \qquad \underline{b} \notin \mathcal{C}(A)$

$A^T A\tilde{\underline{x}} = A^T\underline{b}$ ← Normal equations

$A = \bar{U}\bar{\Sigma}\bar{V}^T$

$$\bar{V}\bar{\Sigma}^T\bar{U}^T\bar{U}\bar{\Sigma}\bar{V}^T\tilde{\underline{x}} = \bar{V}\bar{\Sigma}^T\bar{U}^T\underline{b}$$

$$\bar{\Sigma}\bar{\Sigma}\bar{V}^T\tilde{\underline{x}} = \bar{\Sigma}\bar{U}^T\underline{b}$$

$$\underbrace{\bar{V}\bar{V}^T}_{P}\tilde{\underline{x}} = \bar{V}\bar{\Sigma}^{-1}\bar{U}^T\underline{b}$$

$$\hat{\underline{x}} = \bar{V}\bar{V}^T\tilde{\underline{x}}$$

$$\hat{\underline{x}} = \underbrace{\bar{V}\bar{\Sigma}^{-1}\bar{U}^T\underline{b}}_{A^\dagger\underline{b}}$$

$\|\underline{b} - A\hat{\underline{x}}\|$ is minimal

$\tilde{\underline{x}} = A^\dagger\underline{b}$ ← one of infinitely many solutions / ls-solution.

Q

$$P = \bar{Q}(\bar{Q}^T\bar{Q})^{-1}\bar{Q}^T = \bar{Q}\bar{Q}^T$$

5

$\underline{A}^{+}\underline{b}$ always picks the shortest vector as solution.

Example:

$$\begin{bmatrix}1&1\\0&0\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Find LS-sol:

$$A^TA=\begin{bmatrix}1&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&1\\0&0\\0&0\end{bmatrix}=\begin{bmatrix}1&1\\1&1\end{bmatrix},\quad A^T\underline{b}=\begin{bmatrix}1&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

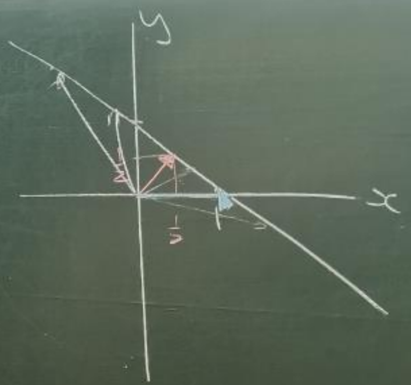
$$\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

$x+y=1$

$y=1-x$

$$\tilde{\underline{x}}=\begin{bmatrix}x\\1-x\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}+x\begin{bmatrix}1\\-1\end{bmatrix}$$

6



ROTATIONS IN 3D:

Cross-P:

$$\underline{a}\times\underline{b}=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\times\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{bmatrix}$$

(*)

7

$$= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= A\underline{b}$$ cp-matrix

Properties of A $(\|\underline{a}\| = 1)$

① Skew symmetric. $A^T = -A$

② A^2 is symmetric.

③ $-A^2$ is a projection matrix

④ $I + A^2 = \underline{a}\,\underline{a}^T$

8

$$A^2 = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{-a_3^2 - a_2^2}\ \ a_1^2-1 & a_1a_2 & a_1a_3 \\ a_1a_2 & \cancel{-a_1^2 - a_3^2}\ \ a_2^2-1 & a_2a_3 \\ a_1a_3 & a_2a_3 & \cancel{-a_1^2 - a_2^2}\ \ a_3^2-1 \end{bmatrix}$$

$$-a_3^2 - a_2^2 + 1 - 1 = \cancel{-a_3^2} \cancel{-a_2^2} + a_1^2 \cancel{+a_2^2} \cancel{+a_3^2} - 1$$

$$= \begin{pmatrix} a_1a_1 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2a_2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3a_3 \end{pmatrix} - I$$

$$A^2 = \underline{a}\,\underline{a}^T - I$$

$$P_{\text{proj on } \underline{a}} = \frac{\underline{a}\,\underline{a}^T}{\cancel{\underline{a}^T\underline{a}}\ 1}$$