

AM214-2023: LECTURE 35

LECTURE 35 SVD (LAST PART), ROTATIONS IN 3D 1

CASE I: $Ax = b$ has a unique solution.

$A = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$ or $\begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$

$A = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} V^T$

$= \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix} \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$
 (Note: Σ is square)

$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$
 Σ^{-1} exists.

$U^T U = I$
 $V^T V = I$
 but $V V^T = I$

$Ax = b$ and $b \in \mathcal{R}(A)$ 2

$U \Sigma V^T x = b$

Pre-m. by U^T :
 $U^T U \Sigma V^T x = U^T b$

Pre-m. by Σ^{-1} :
 $V^T x = \Sigma^{-1} U^T b$

Pre-m. by V :
 $x = V \Sigma^{-1} U^T b$
 $x = A^+ b$

pseudo inverse
 Moore-Penrose inv.
 $\gg \text{pinv}(A)$

DEF: $A^+ = \bar{V} \bar{\Sigma}^{-1} \bar{U}^T$



3

Properties:

$$AA^+A = A$$

$$A^+AA^+ = A^+$$

$$(\bar{U} \bar{\Sigma} \bar{V}^T) (\bar{V} \bar{\Sigma}^{-1} \bar{U}^T) (\bar{U} \bar{\Sigma} \bar{V}^T)$$

$$= \bar{U} \bar{\Sigma} \bar{\Sigma}^{-1} \bar{\Sigma} \bar{V}^T = \bar{U} \bar{\Sigma} \bar{V}^T = A$$

CASE 2. No solution, try LS-solution
 $Ax = b$ $b \notin \mathcal{R}(A)$

$$A^T A \tilde{x} = A^T b \quad \leftarrow \text{Normal equations}$$

$$A = \bar{U} \bar{\Sigma} \bar{V}^T$$

$$\cancel{\bar{V} \bar{\Sigma}^{-1} \bar{U}^T} \bar{U} \bar{\Sigma} \bar{V}^T \tilde{x} = \cancel{\bar{V} \bar{\Sigma}^{-1} \bar{U}^T} \bar{U}^T b$$

$$\cancel{\bar{V} \bar{\Sigma} \bar{V}^T} \tilde{x} = \cancel{\bar{V} \bar{\Sigma} \bar{V}^T} \bar{U}^T b$$

$$\underbrace{\bar{V} \bar{V}^T}_P \tilde{x} = \bar{V} \bar{\Sigma}^{-1} \bar{U}^T b$$

$$\tilde{x} = \bar{V} \bar{V}^T \tilde{x}$$

$$\tilde{x} = \underbrace{\bar{V} \bar{\Sigma}^{-1} \bar{U}^T}_A b$$

$$\|b - A\tilde{x}\| \text{ is minimal}$$

$$\tilde{x} = A^+ b \quad \leftarrow \text{one of infinitely many solutions / LS-solution.}$$

$$P = \bar{Q} \bar{Q}^T$$

$$= \bar{Q} \bar{Q}^T$$

4

$A^+ b$ always picks the shortest vector as solution.

5

Example:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

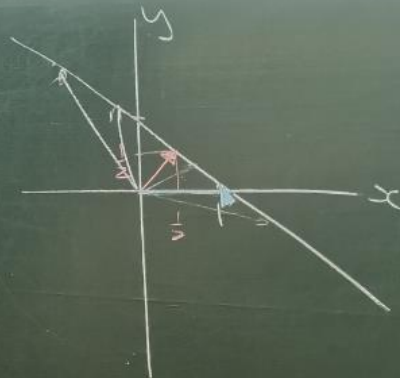
Find LS-sol:

$$A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A^T b = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} x + y &= 1 \\ y &= 1 - x \end{aligned}$$

$$\underline{\tilde{x}} = \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



6

ROTATIONS IN 3D:

Cross-P:

$$\underline{a} \times \underline{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

7

Properties of A ($\|g\|=1$) cp-matrix

- ⊙ Skew symmetric. $A^T = -A$
- ⊙ A^2 is symmetric.
- ⊙ $-A^2$ is a projection matrix
- ⊙ $I + A^2 = gg^T$

$$A^2 = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{a_3^2} \cancel{a_2^2} & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & \cancel{a_3^2} \cancel{a_1^2} & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & \cancel{a_2^2} \cancel{a_1^2} \end{bmatrix}$$

$$-a_3^2 - a_2^2 + 1 = -\cancel{a_3^2} - \cancel{a_2^2} + a_1^2 + \cancel{a_2^2} + \cancel{a_3^2} - 1$$

$$= \begin{bmatrix} a_1 a_1 & a_1 a_2 & a_1 a_3 \\ a_2 a_1 & a_2 a_2 & a_2 a_3 \\ a_3 a_1 & a_3 a_2 & a_3 a_3 \end{bmatrix} - I$$

$$A^2 = gg^T - I$$

Proj on $g = \frac{gg^T}{\|g\|^2}$

8